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Research Paper

The Characteristic Function Property of Mixture Negative Binomial-Exponential Distribution

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Abstract
This paper introduces a new distribution by mixing the negative binomial distribution and exponential distribution namely negative binomial-exponential (NB-E) distribution. It is given the probability distribution function of NB-E distribution and its characteristic function by using Fourier-Stieltjes transform. In addition we present some properties of characteristic function from NB-E distribution.

Keywords
Negative binomial distribution, exponential distribution, characteristic function, mixture distribution

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1. INTRODUCTION

The negative binomial distribution is an important distribution in statistics. The negative binomial distribution is obtained as a mixture of Poisson and gamma distribution (Pudprommarat et al., 2012), this paper also introduced a new mixture distribution by mixing negative binomial whose probability of success parameter $p = e^{-\lambda}$ and Λ follows exponential distribution. Mixture distribution in various research has been developed by several researchers in term of univariate and multivariate analysis of mixture distribution negative binomial-invers Gaussian (Gomez-Deniz et al., 2008), application of negative binomial-beta distribution (Pudprommarat et al., 2012), mixture negative binomial-general exponential (Aryuyuen and Bodhisuwan, 2013), negative binomial-Crack distribution (Saengthong and Bodhisuwan, 2013), the negative binomial-Erlang distribution with applications (Kongrod et al., 2014), exponential-beta distributions (Nadarajah and Kotz, 2006), negative binomial-Lindley distribution (Zamani and Ismail, 2010) and application negative binomial-Lindley distribution (Shirazi et al., 2017).

Suppose that X has a probability density function $f(x;\lambda)$ and if parameter Λ is an absolutely continuous random variable having probability density function $f(\lambda)$ then we will have probability density function of mixture distribution as follows (Lukacs, 1992)

$$f_X(x) = \int_{\forall\lambda} f_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda) d\lambda \tag{1}$$

Let us define X as a random variable that has a negative binomial distribution with parameter $r > 0$ and $p = e^{-\lambda}$ where Λ is a random variable that has an exponential distribution with parameter θ, denoted $X|\lambda \sim \text{NB}(r, p = e^{-\lambda})$ and $\lambda \sim E(\theta)$ then random variable X called random variable of negative binomial-exponential (NB-E) distribution with parameter r and θ, denoted by $X \sim \text{NB-E}(r, p = e^{-\lambda})$.

In the present paper, we study the characteristic of NB-E distribution, characteristic function of mixture distribution in another research has been developed on scale mixtures of multivariate skew-normal distributions (Kim and Genton, 2011) and Rayleigh mixture distribution (Karim et al., 2011). However, it will be discussed in characteristic function and the property of characteristic function of NB-E distribution. In Section 2, we will explain about probability density function of mixture NB-E distribution. While some properties mixture NB-E distribution will be given in Section 3.

2. THE PROBABILITY DENSITY FUNCTION OF MIXTURE NEGATIVE BINOMIAL-EXPONENTIAL DISTRIBUTION

The negative binomial distribution is a family of discrete probability distribution with parameter r dan p. Let X be a random variable from negative binomial distribution with the form (Forbes et al., 2011)

$$f_X(x; r, p) = \binom{r + x - 1}{r} p^r (1 - p)^x \tag{2}$$

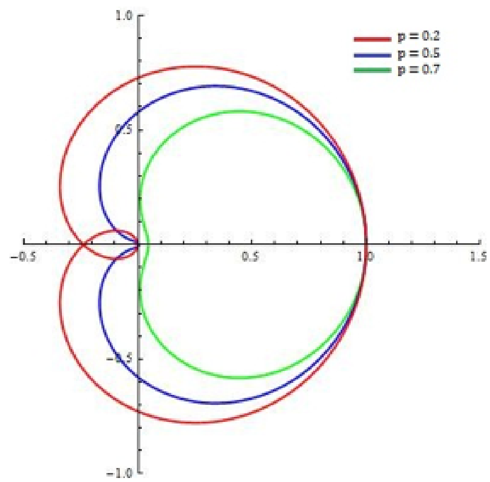

Figure 1. Parametric curves of characteristic function from negative binomial distribution with $r = 5$ and various parameters $p = 0.2, 0.5, 0.7$

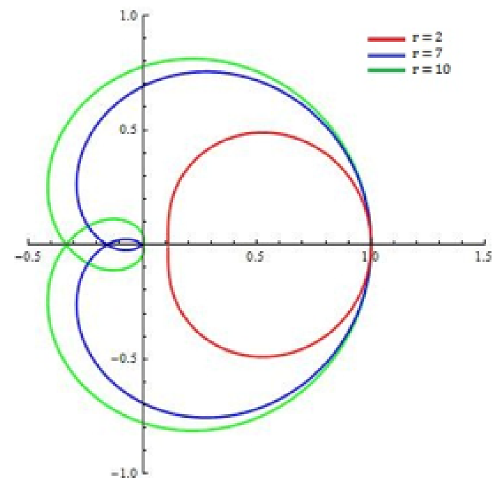

Figure 2. Parametric curves of characteristic function from negative binomial distribution with $p = 0.5$ and various parameters $r = 2, 7, 10$

for $x = 0, 1, 2, 3 \ldots, r > 0$ and $0 \leq p \leq 1$. The expectation and variance of negative binomial distribution are given by

$$E_X(X) = \frac{r(1-p)}{p} \tag{3}$$

$$Var_X(X) = \frac{r(1-p)}{p^2} \tag{4}$$

As for the moment generating function and characteristic function of the negative binomial distribution in the following forms

$$M_X(t) = E\left[exp\,(tX)\right] = \left(\frac{p}{(1-(1-p)e^t)}\right)^r \tag{5}$$

$$\varphi_X(t) = E\left[exp\,(itX)\right] = \left(\frac{p}{(1-(1-p)e^{it})}\right)^r . \tag{6}$$

The Figure 1 is the graph of parametric curves of characteristic function from negative binomial distribution with various parameter r. While the Figure 2 is the graph of parametric curves of characteristic function from negative binomial distribution with various parameter p. The graph of these parametric curves of characteristic function are described in the complex plane that shows a smooth line, this is to confirm that its characteristic function is continuous and never vanish on the complex plane.

The exponential distribution has probability density function as follows Klugman et al. (2012)

$$f_X\,(x;\theta) = \theta e^{-\theta x} \tag{7}$$

for $0 < x < \infty$ and $\theta > 0$. The k moment, expectation, and variance of exponential as follows

$$E_X\left(x^k\right) = \frac{k!}{\theta^k}, \tag{8}$$

$$E_X\,(x) = \frac{1}{\theta}, \tag{9}$$

$$Var_X\,(x) = \frac{1}{\theta^2}, \tag{10}$$

We have the moment generating function and characteristic function of the exponential distribution respectively as follows

$$M_X\,(t) = \left(1 - \frac{t}{\theta}\right)^{-1} \tag{11}$$

$$\varphi_X\,(t) = \left(1 - \frac{it}{\theta}\right)^{-1} \tag{12}$$

The Figure 3 is the graph of parametric curves of characteristic function from exponential distribution with various parameter θ. The graph of its characteristic function is described in the complex plane that shows a smooth line, this is also to confirm that this characteristic function is continuous and never vanish on the complex plane.

Definition 1. *Let X be a random variable of a NB-E (r, θ) distribution, $X \sim NB\text{-}E(r, \theta)$, when the negative binomial distribution have parameter $r > 0$ and $p = e^{-\lambda}$, where λ is distributed as exponential distribution with parameter $\theta > 0$, i.e., $X|\lambda \sim NB(r, p = e^{-\lambda})$ and $\lambda \sim E(\theta)$.*

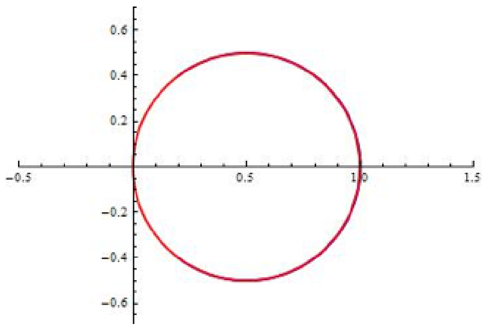

Figure 3. Parametric curves of characteristic function from exponential distribution with various parameters $\theta = 0.5, 50, 1000$

Theorem 1. *Let $X \sim NB\text{-}E(r,\theta)$. The probability density function of X is given by*

$$f_X(x;r,\theta) = \binom{r+x-1}{x} \sum_{n=0}^{x} \binom{x}{n} (-1)^n \left(1 + \frac{(r+n)}{\theta}\right)^{-1} \quad (13)$$

where $x = 0, 1, 2, \ldots$ and $r, \theta > 0$.

Proof. If $X|\lambda \sim NB\left(r, p = e^{-\lambda}\right)$ and $\lambda \sim E(\theta)$, the probability density function of X can be obtained by using

$$f_X(x;r,\theta) = \int_0^\infty f_{X|\Lambda}(x|\lambda) f_\Lambda(\lambda;\theta) d\lambda \quad (14)$$

where $f_{X|\Lambda}(x|\lambda)$ is defined by

$$f_{X|\Lambda}(x|\lambda) = \binom{r+x-1}{r} e^{-\lambda r}\left(1-e^{-\lambda}\right)^x \quad (15)$$

It is used the series expansion (Abramowitz and Stegun, 1972) so that we have

$$\left(1-e^{-\lambda}\right)^x = \sum_{n=0}^{x}\binom{x}{n}(-e^{-\lambda})^n = \sum_{n=0}^{x}\binom{x}{n}(-1)^n e^{-\lambda n} \quad (16)$$

According to Equation (16) we can rewrite Equation (15) as

$$f_{X|P}(x|p) = \binom{r+x-1}{r}\sum_{n=0}^{x}\binom{x}{n}(-1)^n e^{-\lambda(r+n)} \quad (17)$$

By substituting Equation (17) into Equation (14) we obtain

$$\begin{aligned} f_X(x;r,\theta) &= \binom{r+x-1}{r}\sum_{n=0}^{x}\binom{x}{n}(-1)^n\int_0^\infty e^{-\lambda(r+n)} f_\Lambda(\lambda;\theta)d\lambda \\ &= \binom{r+x-1}{r}\sum_{n=0}^{x}\binom{x}{n}(-1)^n M_\lambda\left(-(r+n)\right) \quad (18) \end{aligned}$$

Substituting the moment generating function of exponential distribution in Equation (11) to Equation (18), the probability density function of NB-E(r,θ) is given as

$$f_X(x;r,\theta) = \binom{r+x-1}{x}\sum_{n=0}^{x}\binom{x}{n}(-1)^n\left(1+\frac{(r+n)}{\theta}\right)^{-1} \quad (19)$$

□

3. THE CHARACTERISTIC FUNCTION OF MIXTURE NEGATIVE BINOMIAL-EXPONENTIAL DISTRIBUTION

The characterization of NB-E distribution will be stated on its characteristic function and continuity property at the following theorems and propositions.

Theorem 2. *If X is a random variable NB-E distribution then the characteristic function of X is described as follows*

$$\varphi_X(t) = \frac{\theta}{(r+\theta)(1-e^{it})^r}\, {}_2F_1\left(r+\theta+1, r, r+\theta+2; \frac{e^{it}}{e^{it}-1}\right) \quad (20)$$

where $r, \theta > 0$

Proof. If $X|\lambda \sim$ NB-E$(r, p = e^{-\lambda})$ and $\lambda \sim E(\theta)$, the characteristic function of X can be obtained by using Fourier-Stieltjes transform (see Lukacs (1992))

$$\varphi_X(t) = \int_0^\infty \varphi_{X|\Lambda}(t) f_\Lambda(\lambda;\theta)d\lambda \quad (21)$$

where $\varphi_{X|\Lambda}(x|\lambda)$ is defined by:

$$\varphi_{X|\Lambda}(t) = \left(\frac{e^{-\lambda}}{(1-(1-e^{-\lambda})e^{it})}\right)^r \quad (22)$$

and $f_\Lambda(\lambda;\theta)$ is defined by Equation (7), then Bain and Engelhardt (1992) have introduced the form to rewrite $f_\Lambda(\lambda;\theta)$ on the following term

$$f_\Lambda(\lambda;\theta) = \theta e^{-\theta\lambda}, \quad (23)$$

Next, by substituting Equation (22) and Equation (23) into Equation (21), we obtain

$$\begin{aligned} \varphi_X(t) &= \int_0^\infty \left(\frac{e^{-\lambda}}{(1-(1-e^{-\lambda})e^{it})}\right)^r \theta e^{-\theta\lambda} d\lambda \\ &= \theta\int_0^1 e^{-r\lambda}\left(1-(1-e^{-\lambda})e^{it}\right)^{-r} e^{-\theta\lambda}\frac{de^{-\lambda}}{e^{-\lambda}} \\ &= \theta\int_0^1 e^{-\lambda(r+\theta-1)}\left(1-(1-e^{-\lambda})e^{it}\right)^{-r}\left(\frac{1-e^{it}}{1-e^{it}}\right)^{-r} de^{-\lambda} \\ &= \frac{\theta}{(1-e^{it})^r}\int_0^1 e^{-\lambda(r+\theta-1)}\left(\frac{1-(1-e^{-\lambda})e^{it}}{1-e^{it}}\right)^{-r} de^{-\lambda} \\ &= \frac{\theta}{(1-e^{it})^r}\int_0^1 e^{-\lambda(r+\theta-1)}\left(\frac{1-e^{it}+e^{-\lambda}e^{it}}{1-e^{it}}\right)^{-r} de^{-\lambda} \\ &= \frac{\theta}{(1-e^{it})^r}\int_0^1 e^{-\lambda(r+\theta-1)}\left(1-e^{-\lambda}\left(\frac{e^{it}}{e^{it}-1}\right)\right)^{-r} de^{-\lambda} \end{aligned} \quad (24)$$

It is used definition of Gaussian hypergeometric as in Johnson et al. (2005) in the following terms

$${}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du$$

$$\frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)}\, {}_2F_1(a,b,c;x) = \int_0^1 u^{a-1}(1-u)^{c-a-1}(1-xu)^{-b}du$$

(25)

Equation (24) can be transformed by using Equation (25) then we have

$$\varphi_X(t) = \frac{\theta}{(1-e^{it})^r}\frac{\Gamma(r+\theta)\Gamma(1)}{\Gamma(r+\theta+1)}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{it}}{e^{it}-1}\right) = \frac{\theta}{(r+\theta)(1-e^{it})^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{it}}{e^{it}-1}\right) \quad (26)$$

□

Corollary 3. *If $r = 1$ then characteristic function of NB-E distribution is characteristic function of geometric-exponential (G-E) distribution as follows*

$$\varphi_X(t) = \frac{\theta}{(1+\theta)(1-e^{it})^r}\,{}_2F_1\left(1+\theta, 1, \theta+2, \frac{e^{it}}{e^{it}-1}\right) \quad (27)$$

where $\theta > 0$

Proof. If $X|\lambda \sim G(p = e^{-\lambda})$ and $\lambda \sim E(\theta)$, the characteristic function of X can be obtained by using

$$\varphi_X(t) = \int_0^\infty \varphi_{X|\Lambda}(t) f_\Lambda(\lambda;\theta)d\lambda \quad (28)$$

where $\varphi_{X|\Lambda}(x|\lambda)$ is defined by

$$\varphi_{X|\Lambda}(t) = \frac{e^{-\lambda}}{(1-(1-e^{-\lambda})e^{it})} \quad (29)$$

and $f_\Lambda(\lambda;\theta)$ is define by Equation (7), then we can rewrite by

$$f_\Lambda(\lambda;\theta) = \theta e^{-\theta\lambda}, \quad (30)$$

Now, it is substituting Equation (29) and Equation (30) into Equation (28), we obtain

$$\begin{aligned}\varphi_X(t) &= \int_0^\infty \frac{e^{-\lambda}}{(1-(1-e^{-\lambda})e^{it})}\theta e^{-\theta\lambda}d\lambda \\ &= \theta\int_0^1 e^{-\lambda}\left(1-(1-e^{-\lambda})e^{it}\right)^{-1} e^{-\theta\lambda}\frac{de^{-\lambda}}{e^{-\lambda}} \\ &= \theta\int_0^1 e^{-\lambda(r+\theta-1)}\left(1-(1-e^{-\lambda})e^{it}\right)^{-1}\left(\frac{1-e^{it}}{1-e^{it}}\right)^{-1} de^{-\lambda} \\ &= \frac{\theta}{1-e^{it}}\int_0^1 e^{-\lambda(r+\theta-1)}\left(\frac{1-(1-e^{-\lambda})e^{it}}{1-e^{it}}\right)^{-1} de^{-\lambda} \\ &= \frac{\theta}{1-e^{it}}\int_0^1 e^{-\lambda\theta}\left(\frac{1-e^{it}+e^{-\lambda}e^{it}}{1-e^{it}}\right)^{-1} de^{-\lambda} \\ &= \frac{\theta}{1-e^{it}}\int_0^1 e^{-\lambda\theta}\left(1-e^{-\lambda}\left(\frac{e^{it}}{e^{it}-1}\right)\right)^{-1} de^{-\lambda} \quad (31)\end{aligned}$$

Equation (31) can be transformed by using Equation (25) then we have

$$\varphi_X(t) = \frac{\theta}{(1-e^{it})^r}\frac{\Gamma(r+\theta)\Gamma(1)}{\Gamma(r+\theta+1)}\,{}_2F_1\left(1+\theta, 1, \theta+2, \frac{e^{it}}{e^{it}-1}\right) = \frac{\theta}{(1+\theta)(1-e^{it})}\,{}_2F_1\left(1+\theta, 1, \theta+2, \frac{e^{it}}{e^{it}-1}\right) \quad (32)$$

□

Proposition 4. *Let $\varphi_X(t)$ be a characteristic function of the random variable X with NB-E distribution then $\varphi_X(0) = 1$*

Proof. According to the Theorem 2 we can write characteristic function of the random variable X as follows

$$\varphi_X(t) = \frac{\theta}{(r+\theta)(1-e^{it})^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{it}}{e^{it}-1}\right) \quad (33)$$

Now, we can rewrite Equation (33) as follows

$$\varphi_X(t) = \theta\int_0^1 e^{-\lambda(r+\theta-1)}\left(1-(1-e^{-\lambda})e^{it}\right)^{-r} de^{-\lambda} \quad (34)$$

Base on the Equation (34) if $t = 0$ then we have

$$\varphi_X(0) = 1 \quad (35)$$

□

Proposition 5. *If X is a random variable NB-E distribution then the characteristic function of X is continuous*

Proof. The continuity property is explained by using definition of uniform continuity, that is for every $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varphi_X(s) - \varphi_X(t)| < \varepsilon$ for $|s-t| < \delta$ where δ depends only on ε. Then the uniform continuity of characteristic function is obtained by the following way. First we write the following equation

$$\begin{aligned}|\varphi_X(s) - \varphi_X(t)| &= \left|\frac{\theta}{(r+\theta)(1-e^{is})^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{is}}{e^{is}-1}\right) - \frac{\theta}{(r+\theta)(1-e^{it})^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{it}}{e^{it}-1}\right)\right| \\ &= \left|\frac{\theta}{(r+\theta)}\left[\frac{1}{(1-e^{is})^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{is}}{e^{is}-1}\right) - \frac{1}{(1-e^{it})^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{it}}{e^{it}-1}\right)\right]\right| \quad (36)\end{aligned}$$

Now let us define $h = s - t$, so that for $h \to 0$ we have the following limit

$$\lim_{h\to 0}\left[\frac{1}{\left(1-e^{i(h+t)}\right)^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{i(h+t)}}{e^{i(h+t)}-1}\right) - \frac{1}{\left(1-e^{it}\right)^r}\,{}_2F_1\left(r+\theta, r, r+\theta+1, \frac{e^{it}}{e^{it}-1}\right)\right] = 0. \quad (37)$$

This hold for $\delta < \varepsilon$ where $|\varphi_X(h+t) - \varphi_S(t)| < \varepsilon$ for $|s-t| < \delta$. Then $\varphi_X(t)$ is uniformly continuous.

Based on the graphs presented in Figures 4 and 5, it can be concluded that the characteristic function of NB-E distribution is continuous, as well as it has discussed on Proposition 5. In Figure 4 and 5, the shape of parametric curves form three smooth lines and never vanish on the complex plane. □

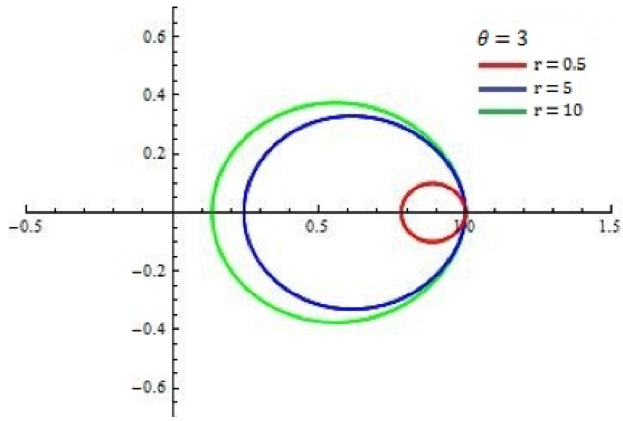

Figure 4. Parametric curves of characteristic function from NB-E distribution with various parameters $r = 0.5, 5, 10$ where $\theta = 3$

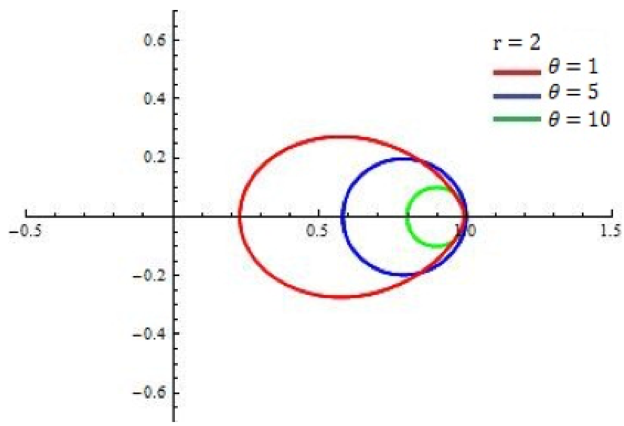

Figure 5. Parametric curves of characteristic function from NB-E distribution with various parameters $\theta = 1, 5, 10$ where $r = 2$

4. CONCLUSIONS

The negative binomial-exponential (NB-E) distribution is the mixture negative binomial distribution and exponential distribution. The characterization of NB-E distribution is obtained by using the property of characteristic function, where the characteristic function is defined as the Gaussian hypergeometric function. The characteristic function of this mixture NB-E distribution has the continuity property, where the graph of the parametric curves of characteristic function is in the smooth line and never vanish on the complex plane.

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