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Research Paper

Properties and Generalizations of Altered Jacobsthal Numbers Squared and their GCD Sequences

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Abstract
This paper investigates two types of altered Jacobsthal numbers, namely $G^{(2)}_{J(n)}(a)$ and $H^{(2)}_{J(n)}(a)$, which are obtained by adding or subtracting a specific value, denoted with $\{a\}$, from the square of the n^{th} Jacobsthal numbers. These numbers exhibit a close relationship with the consecutive products of the Jacobsthal numbers. The study establishes consecutive sum-subtraction relations for the altered Jacobsthal numbers, and derives their Binet-like formulas. Furthermore, the greatest common divisor (Gcd) sequences of r-successive terms, represented by $\{G^{(2)}_{J(n),r}(a)\}$ and $\{H^{(2)}_{J(n),r}(a)\}$, $r \in \{1, 2, 3, 4\}$ are investigated. It is observed that these sequences display either a periodic or Jacobsthal structure.

Keywords
Altered Jacobsthal Number, Greatest Common Divisor (GCD) Sequence, Jacobsthal Sequence

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1. INTRODUCTION

Rule 28 is recognized as one of the fundamental rules governing elementary cellular automata, initially introduced by Wolfram (1983). This rule determines the subsequent state of a cell based on its current state and the states of its immediate neighbors. The binary encoding of Rule 28 is given by $28 = (00011100)_2$. A significant simplification in computing the n^{th} generation under Rule 28 is achieved by reducing the initial configuration to a single black cell (Wolfram and Gad-el Hak, 2003; Wolfram, 1983). This simplification leverages the Equation (1):

$$J_n = \frac{2^n - (-1)^n}{3}, \quad n \geq 2, \tag{1}$$

where J_n denotes the n^{th} Jacobsthal number. The identity in Equation 1 is analogous to a Binet-type formula. It is well-established that the Jacobsthal numbers J_n and the Jacobsthal-Lucas numbers j_n are defined recursively through a second-order recurrence relation given by Equation (2).

$$X_n = X_{n-1} + 2X_{n-2}, \quad n \geq 2, \tag{2}$$

with initial conditions $J_0 = 0$, $J_1 = 1$, and $j_0 = 2$, $j_1 = 1$. Furthermore, the n^{th} Jacobsthal-Lucas number can be represented using the Binet-like expression in Equation (3):

$$j_n = 2^n + (-1)^n, \quad n \in \mathbb{Z}. \tag{3}$$

The Binet-like expressions provided in Equations 1 and 3 serve as explicit identities that facilitate the derivation of various properties associated with these sequences (Horadam, 1988, 1993). For instance, these formulas can be extended to accommodate negative indices, leading to the identities $J_{-n} = \frac{(-1)^{n+1}}{2^n} J_n$ and $j_{-n} = \frac{(-1)^n}{2^n} j_n$.The Jacobsthal numbers hold particular significance in computational analysis, especially in evaluating the time complexity of algorithms. They can be employed to determine the number of iterations required at various stages of an algorithm. Moreover, these numbers are instrumental in optimizing the performance of data structures. For example, the sequence of Jacobsthal numbers aids in calculating the number of iterations at different algorithmic steps, as detailed in (Horadam, 1988, 1993) and cataloged as sequence A001045 in the Online Encyclopedia of Integer Sequences (Sloane, 2003). The sequence identifier, A001045, can be used as a reference in the OEIS database for further exploration.

Horadam (1988) investigated various identities associated with second order Jacobsthal numbers. One such identity is given by Equation (4).

$$J^2_{m+k} + (-1)^m 2^{m-1} J^2_{k+1} = J_{m-1} J_{m+2k+1}, \tag{4}$$

which has been observed in certain applications of Jacobsthal numbers in the study of curves.

Later, Horadam (1993) introduced the k^{th} associated Jacobsthal number, denoted as $J^{(k)}_n$, defined in terms of the

Jacobsthal J_n and Jacobsthal-Lucas j_n numbers as Equation (5):

$$J_n^{(k)} = J_{n+1}^{(k-1)} + 2J_{n-1}^{(k-1)}, \tag{5}$$

with initial conditions $J_n^{(0)} = J_n$ and $J_n^{(1)} = j_n$ for $n, k \geq 1$.

For the case $k = 1$, it follows that: $J_n^{(1)} = J_{n+1}^{(0)} + 2J_{n-1}^{(0)}$. Since $J_n^{(0)} = J_n$ and the identity $J_{n+1} + 2J_{n-1} = j_n$ hold, it follows that $J_n^{(1)} = j_n$, thereby verifying the identity. Similarly, for $k = 2$, the relationship becomes: $J_n^{(2)} = J_{n+1}^{(1)} + 2J_{n-1}^{(1)}$. Given $J_n^{(1)} = j_n$ and the identity $j_{n+1} + 2j_{n-1} = 9J_n$, it follows that $J_n^{(2)} = 3^2 J_n$. Using mathematical induction, Horadam demonstrated the general relationships: $J_n^{(2m)} = 3^{2m} J_n$, $J_n^{(2m+1)} = 3^{2m} j_n$. In addition, Horadam defined a sequence denoted by $\hat{J}_n$ (cataloged as A000975 in Sloane (2003) and discussed in Horadam (1996b), using the summation formula for consecutive Jacobsthal numbers in Equation (6):

$$\hat{J}_n = \sum_{i=2}^{n} J_i = \frac{J_{n+2} - 3}{2}, \quad \hat{J}_0 = 0, \ \hat{J}_1 = 1. \tag{6}$$

Furthermore, numerous properties of the associated Jacobsthal numbers $J_n^{(k)}$, as defined in Horadam (1993), were established. Several additional properties of the derived sequence $\hat{J}_n^{(k)}$, based on the definition in Equation (6), were also explored by leveraging the properties of the sequence $\hat{J}_n$.

In addition to the sequences defined in Equations (5-6), (Horadam, 1996a), examined the recurrence relation:

$$\Upsilon_{n+2}(a, b, k) = \Upsilon_{n+1} + 2\Upsilon_n + k, \quad \Upsilon_0 = a, \ \Upsilon_1 = b,$$

where a, b, k are arbitrary integers. Specific instances of the sequence $\Upsilon_n(a, b, k)$ and their corresponding values were provided, including: $\Upsilon_{2n}(1, 1, 1) = 3J_{2n} + 1 = 2^{4n}$, $\Upsilon_{2n+1}(1, 1, 1) = 3J_{2n+1} - 2 = 2^{4n+2} - 1$, $\Upsilon_{2n}(1, 2, 0) = 2^n$.

Catarino et al. (2015) introduced a novel number, $J_n^{(k)}$, distinct from the associated numbers defined in Equation (5). This new number was expressed as Equation (7):

$$J_n^{(k)} = \frac{\left(r_1^{m+1} - r_2^{m+1}\right)^r \left(r_1^m - r_2^m\right)^{k-r}}{(r_1 - r_2)^k}, \quad n = mk + r,$$
$$r_1 = 2, \ r_2 = -1, \tag{7}$$

where $J_n^{(k)}$ generalizes the classical Jacobsthal numbers J_n. The authors also provided generating functions for specific cases of the numbers defined in Equation (7).

In Uygun and Owusu (2016) and Uygun (2021), a bi-periodic Jacobsthal sequence was introduced. This sequence is governed by the recurrence relation:

$$J_n = \begin{cases} aJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is even,} \\ bJ_{n-1} + 2J_{n-2}, & \text{if } n \text{ is odd,} \end{cases} \quad J_0 = 0, \ J_1 = 1, \ n \geq 2,$$

where a and b are nonzero real numbers. The authors also derived new identities for this bi-periodic Jacobsthal sequence. Yazlik et al. (2016) explored the sets of remainders of Jacobsthal numbers modulo m. Their study provided significant insights into the properties of these sets and introduced an innovative method for determining the length of the periodicity modulo m.

Koken (2019) defined the altered Jacobsthal sequences, denoted by $\{J_n^+\}_{n\geq1}$ and $\{J_n^-\}_{n\geq1}$, which are derived from the Jacobsthal numbers by the following modifications:

$$J_n^+ = \begin{cases} J_n + 2^{\frac{n}{2}-1}, & \text{if } n \text{ is even,} \\ J_n - 2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd,} \end{cases}$$
$$J_n^- = \begin{cases} J_n - 2^{\frac{n}{2}-1}, & \text{if } n \text{ is even,} \\ J_n + 2^{\frac{n-1}{2}}, & \text{if } n \text{ is odd.} \end{cases} \tag{8}$$

The altered Jacobsthal numbers J_n^+ and J_n^-, as given in Equation (8), satisfy identities in Equations (9-10).

$$J_{4k}^+ = j_{2k+1}J_{2k-1}, \ J_{4k+1}^+ = j_{2k+1}J_{2k}, \ J_{4k+2}^+ = J_{2k+2}j_{2k},$$
$$J_{4k+3}^+ = J_{2k+2}j_{2k+1} \tag{9}$$

$$J_{4k}^- = J_{2k+1}j_{2k-1}, \ J_{4k+1}^- = J_{2k+1}j_{2k}, \ J_{4k+2}^- = j_{2k+2}J_{2k},$$
$$J_{4k+3}^- = j_{2k+2}J_{2k+1} \tag{10}$$

The author also examined recurrence relations, Binet-like formulas, Cassini-like identities, explicit expressions, and other significant findings related to the altered Jacobsthal sequences.

Several sequences associated with the altered Jacobsthal numbers are available in the Sloane (2003), including formulas like Equations (9-10). For instance, the Jacobsthal Oblong number, defined as $J_{O_n} = J_n J_{n+1}$, is cataloged as A084175 in Sloane (2003). The studies of integer sequences have been a central focus for many researchers. Their unique properties and interrelations have been extensively analyzed. For example, Wamiliana et al. (2019) examined the sum of the cubes of Lucas and generalized Fibonacci numbers.

Catarino and Borges (2019) introduced the Leonardo numbers $\{L_{e_n}\}_{n\geq0}$, which satisfy the recurrence relation: $L_{e_n} = L_{e_{n-1}} + L_{e_{n-2}} + 1$, $n \geq 2$, with initial conditions $L_{e_0} = L_{e_1} = 1$. Subsequently, Bensella and Behloul (2024) investigated the common terms between the Leonardo sequence $\{L_{e_n}\}_{n\geq0}$ and the Jacobsthal sequence $\{J_m\}_{m\geq0}$. By employing Baker's theory on linear forms in logarithms of algebraic numbers, alongside a variant of the Baker-Davenport reduction method, the authors addressed the Diophantine equation $L_{e_n} = J_m$.

In Brod (2020) and Bilgici and Bród (2023), a new Jacobsth al-type sequence $J_{r,n}$ was introduced, defined by the recurrence relation Equation (11):

$$J_{r,n} = 2^r J_{r,n-1} + (2^r + 4^r) J_{r,n-2}, \ J_{r,0} = 0,$$
$$J_{r,1} = 1 + 2^{r+1}, \ n \geq 2. \tag{11}$$

The authors analyzed various properties of the sequence $J_{r,n}$ as described in Equation (11).

Erduvan and Keskin (2021) investigated the Fibonacci numbers that can be expressed as the product of two Jacobsthal numbers, as well as Jacobsthal numbers that can be represented as the product of two Fibonacci numbers. Their work established general relationships between the n^{th} Fibonacci number F_n and the Jacobsthal number J_n.

The Catalan numbers, often used in combinatorics for counting and enumeration, are defined by: $C_n = \frac{1}{n+1}\binom{2n}{n}$. Yuliana (2023) discussed the construction of multisets and their relationships with Stirling, Bell, and Catalan numbers.

Komatsu and Pita-Ruiz (2023) introduced a formula for calculating the largest integer, known as the p-Frobenius number, allowing limited representations of linear equations involving a Jacobsthal triplet. Additionally, they derived a closed formula for computing the count of non-negative integers, termed the p-genus.

2. EXPERIMENTAL SECTION

This section introduces and analyzes two types of altered Jacobsthal numbers, $G^{(2)}_{J(n)}(a)$ and $H^{(2)}_{J(n)}(a)$, which are derived by adding or subtracting a parameter a to the square of the n^{th} Jacobsthal number, based on whether the index of the altered number is even or odd, respectively. These numbers are explicitly expressed in terms of successive products of Jacobsthal numbers. The derivation of these formulas utilizes established identities of Jacobsthal numbers, including their Binet-like expressions and recurrence relations. Furthermore, it examines the greatest common divisor (Gcd) sequences derived from consecutive altered Jacobsthal numbers, denoted as $G^{(2)}_{J(n),r}(a)$ and $H^{(2)}_{J(n),r}(a)$ for r-successive terms. These Gcd sequences are shown to exhibit periodic or structured behaviors, highlighting the strong divisibility and modularity properties inherent to Jacobsthal numbers.

2.1 Altered Jacobsthal Numbers and Their Properties

Inspired by the identity given in Equation (4), we present two equations that are pertinent to our research objective. Specifically, we consider the sum and subtraction formulas involving the squares of Jacobsthal numbers in Equations (12-13).

$$J^2_{m+k-1} + 2^{2k-1} J^2_{m-k} = J_{2m-1} J_{2k-1}, \tag{12}$$

$$J^2_{m+k} - 2^{2k} J^2_{m-k} = J_{2m} J_{2k}, \quad m > k \tag{13}$$

Here, m and k represent positive integers. Although these equations in Equations (12-13) have not been previously recognized in the literature, we can establish their validity by employing the Binet-like formula described in Equation (1).

Definition 1. The n^{th} altered Jacobsthal numbers, denoted as $G^{(2)}_{J(n)}(a)$ and $H^{(2)}_{J(n)}(a)$ are defined as Equations (14-15).

$$G^{(2)}_{J(n)}(a) = J^2_n + (-1)^n a \tag{14}$$

$$H^{(2)}_{J(n)}(a) = J^2_n - (-1)^n a \tag{15}$$

where J_n be n^{th} Jacobsthal number and $a \in Z$.

For instance, when a is 2^{n-1} and 2^{n-2}, respectively, the numbers $G^{(2)}_{J(n)}(2^{n-1})$ and $H^{(2)}_{J(n)}(2^{n-2})$ in Equations (14-15) form increasing sequences with notable values, except for the initial few values. The general terms of these sequences can be expressed as follows.

Theorem 1. Let $G^{(2)}_{J(n)}(2^{n-1})$ and $H^{(2)}_{J(n)}(2^{n-2})$ denote the n^{th} altered Jacobsthal numbers. Then, the following statements hold Equations (16-17):

$$G^{(2)}_{J(n)}\left(2^{n-1}\right) = J_{n+1} J_{n-1} \tag{16}$$

$$H^{(2)}_{J(n)}\left(2^{n-2}\right) = J_{n+2} J_{n-2} \tag{17}$$

Proof. If we take values of $m = u + 1$ and $k = u$ in Equations (12) and (13), then the number $G^{(2)}_{J(n)}(2^{n-1})$ is respectively obtained according to $n = 2u$ and $n = 2u + 1$ for $a = 2^{n-1}$, by using identity given in Equation (14) as

$$\begin{aligned} G^{(2)}_{J(2u)}\left(2^{2u-1}\right) &= J^2_{2u} + 2^{2u-1} = J_{2u+1} J_{2u-1}, \\ G^{(2)}_{J(2u+1)}\left(2^{2u}\right) &= J^2_{2u+1} - 2^{2u} = J_{2u+2} J_{2u}. \end{aligned}$$

If we use values of $m = u + 2$ and $n = u$ in Equations (12) and (13), then for $a = 2^{n-2}$, by using identity in Equation (15), the number $H^{(2)}_{J(n)}(2^{n-2})$ is respectively achieved as according to both n is odd and n is even

$$\begin{aligned} H^{(2)}_{J(2u+1)}\left(2^{2u-1}\right) &= J^2_{2u+1} + 2^{2u-1} = J_{2u+3} J_{2u-1}, \\ H^{(2)}_{J(2u+2)}\left(2^{2u}\right) &= J^2_{2u+2} - 2^{2u} = J_{2u+4} J_{2u}. \end{aligned}$$

Now, let's research on some sum and subtraction identities of the numbers $G^{(2)}_{J(n)}(2^{n-1})$ and $H^{(2)}_{J(n)}(2^{n-2})$ according to identities given in Equations (16) and (17).

Theorem 2. Let X_n be the n^{th} altered Jacobsthal numbers $G^{(2)}_{J(n)}(2^{n-1})$ and $H^{(2)}_{J(n)}(2^{n-2})$. Then, the following identities hold Equations (18-20):

$$X_{n+1} + 2X_n = J_{2n+1} \tag{18}$$

$$X_{n+1} - 4X_{n-1} = J_{2n} \tag{19}$$

$$X_{n+2} = 3X_{n+1} + 6X_n - 8X_{n-1} \tag{20}$$

Proof. According to the identities in Equations (16) and (17), if we rewrite two from equations in Equations (18) and (19) by using the identities $J_{n+1}^2 + 2J_n^2 = J_{2n+1}$, $2J_{n-1} + J_{n+1} = j_n$ and $J_n j_n = J_{2n}$, we have

$$\begin{aligned} H_{J(n+1)}^{(2)}\left(2^{n-2}\right) + 2H_{J(n)}^{(2)}\left(2^{n-2}\right) &= \left(J_{n+2} + 2J_{n+1}\right) J_{n-1} + 2J_{n+2}J_{n-2} \\ &= \left(J_{n+1} + 2J_n\right) J_n + 2J_{n+1} J_{n-1} \\ &= J_{2n+1} \end{aligned}$$

$$\begin{aligned} G_{J(n+1)}^{(2)}\left(2^{n-1}\right) - 4G_{J(n-1)}^{(2)}\left(2^{n-1}\right) &= \left(J_{n+2} - 4J_{n-2}\right) J_n \\ &= \left(J_{n+1} + 2J_{n-1}\right) J_n \\ &= J_{2n} \end{aligned}$$

Since the other relations follow a similar pattern, they are omitted for the sake of brevity.

By multiplying equations in Equations (18) and (19) by the appropriate values and adding them together, we obtain the following identities mentioned in Equation (20).

Consequently, based on the equations presented in Equations (18) and (19), we observe that the recurrance relation in Equation (2) for the numbers specified in Equations (16) and (17) is equal to a Jacobsthal number. The recurrence relation for these altered numbers is given with equation in Equation (20).

Now, by considering a value $a \in \left\{2^{n-t}J_t^2\right\}$, $(t < n)$, we can extend the concept of altered Jacobsthal numbers introduced in Equations (14-15) to $G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ and $H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$.

Theorem 3. Let $G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ and $H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ be the n^{th} altered Jacobsthal numbers. Then, Equations (21-22) assertions are valid:

$$G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = J_{n+t}J_{n-t}, \ if\ t\ is\ odd \tag{21}$$

$$H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = J_{n+t}J_{n-t}, \ if\ t\ is\ even \tag{22}$$

where J_t^2 denotes a square of the t^{th} Jacobsthal number.

Proof. When t is odd, we write $m = u + (t+1)/2$ and $k = u - (t-1)/2$ in Equations (12) and (13), respectively, they are valid:

$$\begin{aligned} J_{2u}^2 + 2^{2u-t}J_t^2 &= J_{2u+t}J_{2u-t} \\ J_{2u+1}^2 - 2^{2u+1-t}J_t^2 &= J_{2u+1+t}J_{2u+1-t}. \end{aligned}$$

Thus, we have $G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = J_{n+t}J_{n-t}$ for $a = 2^{n-t}J_t^2$ in Equation (14) an according to $n = 2u$ and $n = 2u+1$.

Similarly, when t is even, if we write values of $m = u + t/2$ and $k = u - t/2$ in Equations (12) and (13), respectively, then we have

$$\begin{aligned} J_{2u-1}^2 + 2^{2u-t-1}J_t^2 &= J_{2u+t-1}J_{2u-t-1}, \\ J_{2u}^2 - 2^{2u-t}J_t^2 &= J_{2u+t}J_{2u-t}, \end{aligned}$$

for $a = 2^{n-t}J_t^2$ according to $n = 2u - 1$ and $n = 2u$ in Equation (15). It is valid $H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = J_{n+t}J_{n-t}$.

Now, any Binet like formula are achieved for the numbers $G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ and $H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ given in Equations (21) and (22).

Theorem 4. Let $G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ and $H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right)$ be the n^{th} altered Jacobsthal numbers. Then, they can be expressed with Equations (23-24).

$$G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = \frac{\left(2^{2n}+1\right) + (-1)^n\left(2^{2t}+1\right)}{9}, \ if\ t\ is\ odd \tag{23}$$

$$H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = \frac{\left(2^{2n}+1\right) - (-1)^n\left(2^{2t}+1\right)}{9}, \ if\ t\ is\ even \tag{24}$$

Proof. If the Binet-like formula in Equation (1) are substituted in Equations (21-22) , respectively, and they are adjusted, we have the desired results.

Using the Binet-like formula given in Equations (3), (23) and (24), these numbers are associated with the Jacobsthal Lucas numbers as follows:

$$G_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = \frac{j_{2n} + (-1)^n j_{2t}}{9}, \ if\ t\ is\ odd$$

$$H_{J(n)}^{(2)}\left(2^{n-t}J_t^2\right) = \frac{j_{2n} - (-1)^n j_{2t}}{9}, \ if\ t\ is\ even.$$

Also, the Binet-like formula in given Equations (23) and (24) can be used to prove many properties of these numbers.

2.2 $G_{J(n),r}^{(2)}(a)$ and $H_{J(n),r}^{(2)}(a)$ Altered Jacobsthal Gcd Sequences

The exploration of sequences with altered components underscores the consistent regularity of Jacobsthal numbers in product states. A review of the literature highlights significant studies on product states and Gcd properties in integer sequences, summarizing key contributions and their implications.

The Hosoya triangle, akin to Pascal's triangle, features a triangular array where each element is the product of two Fibonacci numbers (Hosoya, 1976; Koshy, 2019; Flórez and Junes, 2012; Flórez et al., 2014b). Due to its structure, it is often called the Fibonacci triangle.

Hoggatt and Hansell (1971) established that the product of all elements in a Star of David configuration of length two within Pascal's triangle forms a perfect square. Hillman and Hogarth extended this, showing that the Gcd of these elements remains invariant (Hillman and Hoggatt, 1972). These are known as the product and Gcd properties of the Star of David.

In the Hosoya triangle, Flórez and Junes (2012) confirmed the universality of both properties for Star of David configurations of length two. Extending this to generalized Fibonacci numbers, Flórez et al. (2014c) analyzed the generalized Hosoya triangle, where entries are products of two generalized Fibonacci numbers. They introduced the generalized Star of David (GSD), derived from hexagonal configurations, demonstrating the Gcd property for length-two configurations and establishing conditions for length-three configurations. Additionally, they examined the Gcd and modularity properties of generalized Fibonacci numbers (Flórez et al., 2014c,a).

Replacing numerical entries in the Hosoya triangle with polynomials results in a Hosoya-like polynomial triangle. Flórez et al. (2018b) extended Hosoya's numerical recurrence relations to polynomials, constructing triangles where entries are products of Fibonacci or Lucas polynomials. They generalized the Star of David property (Hoggatt-Hansell identity) to these triangles and explored geometric patterns, deriving interpretations for identities like Cassini and Catalan.

Koshy (2019) suggested that various triangular arrays exhibit properties analogous to the Star of David, including the Hoggatt-Hansell identity and the Gcd property. These findings were confirmed for Hosoya and generalized Hosoya triangles. Flórez et al. (2018a) further extended these results, generalizing prior work on numerical sequences (Flórez and Junes, 2012; Flórez et al., 2014a,c,b).

It is widely known that the Jacobsthal sequence satisfies the property of strong divisibility, which asserts that the gcd of two Jacobsthal numbers is itself a Jacobsthal number. Specifically, for positive integers a and b, the following holds: $\gcd(J_a, J_b) = J_{\gcd(a,b)}$. Additionally, the Jacobsthal sequence possesses elliptic divisibility properties, which state that if J_n divides J_m, then n divides m. The divisibility rules and the following propositions, adapted from Flórez et al. (2018a) in relation to Jacobsthal numbers, are presented Equations (25-27):

Let $a, b, c,$ and d be positive integers. If $\gcd(J_a, J_b) = 1$ and $\gcd(J_c, J_d) = 1$, then the following propositions hold:

$$gcd(J_aJ_b, J_cJ_d) = J_{gcd(a,c)}J_{gcd(a,d)}J_{gcd(b,d)}J_{gcd(b,c)} \quad (25)$$

If $|a - c| \leq 2$ and $|b - d| \leq 2$, then the following propositions hold: $gcd(J_a, J_c) = 1$, $gcd(J_b, J_d) = 1$ and

$$gcd(J_aJ_b, J_cJ_d) = J_{gcd(a,d)}J_{gcd(b,c)}. \quad (26)$$

If $gcd(J_a, J_c) = x$ and $gcd(J_b, J_d) = y$, then the following proposition hold:

$$gcd(J_aJ_b, J_cJ_d) = \frac{gcd(yJ_a, xJ_d)\, gcd(xJ_b, yJ_c)}{xy}. \quad (27)$$

We investigate certain properties concerning properties of Gcd of two numbers from the altered sequences $\left\{G^{(2)}_{J(n)}(a)\right\}$ and $\left\{H^{(2)}_{J(n)}(a)\right\}$.

Definition. Let $G^{(2)}_{J(n)}(a)$ and $H^{(2)}_{J(n)}(a)$ be the n^{th} altered Jacobsthal numbers in Equations (14-15). Equations (28-29) expressions

$$G^{(2)}_{J(n),r}(a) = gcd\left(G^{(2)}_{J(n)}(a), G^{(2)}_{J(n+r)}(a)\right) \quad (28)$$

$$H^{(2)}_{J(n),r}(a) = gcd\left(H^{(2)}_{J(n)}(a), H^{(2)}_{J(n+r)}(a)\right) \quad (29)$$

are called as the r-successive altered Jacobsthal gcd numbers.

In the subsequent analysis, for the sake of simplicity in the proofs, we will use the notation (a, b) to represent $\gcd(a, b)$, which is the greatest common divisor of a and b.

By considering the values $a = 2^{n-t}$, $t \in \{1, 2\}$ and $r = 1$ in Equations (28) and (29), we observe that the sequences $\left\{G^{(2)}_{J(n),1}(2^{n-1})\right\}$ and $\left\{H^{(2)}_{J(n),1}(2^{n-2})\right\}$ do not exhibit a strictly increasing or decreasing pattern. They manifest periodic behavior, it's possible. Thus we investigate whether the 1-successive altered Jacobsthal gcd sequences assume particular values within specific periods.

Theorem 5. Let $G^{(2)}_{J(n),1}(2^{n-1})$ and $H^{(2)}_{J(n),1}(2^{n-2})$ be the n^{th} *1*-successive altered Jacobsthal *gcd* numbers. Then, the following statements hold:

$$G^{(2)}_{J(n),1}\left(2^{n-1}\right) = \begin{cases} J_3, & n \equiv 1\,(mod\ 3) \\ 1, & otherwise \end{cases}$$

$$H^{(2)}_{J(n),1}\left(2^{n-2}\right) = \begin{cases} J_5J_3, & n \equiv 7\,(mod\ 15) \\ J_5, & n \equiv 2, 12\,(mod\ 15) \\ J_3, & n \equiv 1, 4, 10, 13\,(mod\ 15) \\ 1, & otherwise \end{cases}$$

proof. We rewrite $G^{(2)}_{J(n),1}(2^{n-1}) = (J_{n+1}J_{n-1}, J_nJ_{n+2})$ according to the identity in Equation (16) for $r = 1$ and $t = 1$ in Equation (28). Since $(J_{n+1}, J_{n-1}) = (J_n, J_{n+2}) = 1$, we take as $G^{(2)}_{J(n),1}(2^{n-1}) = (J_{n-1}, J_{n+2})$ by using identity in Equation (25). So, we obtain the situation $G^{(2)}_{J(n),1}(2^{n-1}) = J_{(n-1,n+2)}$ with strong divisibility property $(J_a, J_b) = J_{(a,b)}$. Thus, $J_{(n-1,3)} = J_3$, $n \equiv 1\,(mod\ 3)$, otherwise, $J_{(n-1,3)} = J_1$ by using $(a, b) = (a, b - ax)$.
We consider $H^{(2)}_{J(n),1}(2^{n-2}) = (J_{n+2}J_{n-2}, J_{n+3}J_{n-1})$ according to Equation (17) for $r = 1$ and $t = 2$ in Equation (29). Since $(J_{n+2}, J_{n+3}) = (J_{n-2}, J_{n-1}) = 1$, we take as $H^{(2)}_{J(n),1}(2^{n-2}) = (J_{n-}, J_{n+3})(J_{n+2}, J_{n-1})$, using identity in Equation (26). Thus, if we evaluate as

$$J_{(n-2,n+3)} = J_{(n-2,5)} = J_5, \quad n \equiv 2\,(mod\ 5)$$

$$J_{(n+2,n-1)} = J_{(n-1,3)} = J_3, \quad n \equiv 1\,(mod\ 3),$$

then the other cases are $(J_{n-2}, J_{n+3}) = (J_{n+2}, J_{n-1}) = 1$. In all cases, the desired results are achieved by employing the Chinese remainder theorem.

If we generate the terms of the 2-successive altered gcd sequences using the values $a = 2^{(n-t)}$, $t \in \{1, 2\}$ and $r = 2$, as specified in Equations (28) and (29), we observe the following patterns: The sequence $\left\{G^{(2)}_{J(n),2}(2^{n-1})\right\}$, for $n \geqslant 1$, exhibits a certain increasing sequence of values. On the other hand, the sequence $\left\{H^{(2)}_{J(n),2}\left(2^{n-2}\right)\right\}$, for $n \geqslant 2$, displays periodic behavior.

Theorem 6. Let $G^{(2)}_{J(n),2}\left(2^{n-1}\right)$ and $H^{(2)}_{J(n),2}\left(2^{n-2}\right)$ be the n^{th} *2*-successive altered Jacobsthal gcd numbers.Then, the following statements hold:

$$G^{(2)}_{J(n),2}\left(2^{n-1}\right)=\begin{cases} J_4J_{n+1}, & n\equiv 1\,(mod\ 4)\\ J_{n+1}, & otherwise\end{cases}$$

$$H^{(2)}_{J(n),2}\left(2^{n-2}\right)=\begin{cases} J_6, & n\equiv 2\,(mod\ 6)\\ J_3, & n\equiv 5\,(mod\ 6)\\ 1, & otherwise\end{cases}$$

proof. If $G^{(2)}_{J(n),2}\left(2^{n-1}\right)=J_{n+1}\left(J_{n-1},J_{n+3}\right)$ is expressed in terms of the identity in Equation (16), then we have $J_{(n-1,n+3)}=J_{(n-1,4)}=J_4$, $n\equiv 1\,(mod\ 4)$, otherwise, $J_{(n-1,4)}=J_2$, $n\equiv 3\,(mod\ 4)$ and J_1, $n\equiv 0,2\,(mod\ 4)$. It has been demonstrated that $G^{(2)}_{J(n),2}\left(2^{n-1}\right)=J_4J_{n+1}$, $n\equiv 1\,(mod\ 4)$. Conversely, $G^{(2)}_{J(n),2}\left(2^{n-1}\right)=J_{n+1}$ in the other cases.
By using the identity in Equation (17), it is rewritten in form of $H^{(2)}_{J(n),2}\left(2^{n-2}\right)=\left(J_{n+2}J_{n-2},J_{n+4}J_n\right)$. Since $\left(J_{n+2},J_{n+4}\right)=\left(J_{n-2},J_n\right)=1$, according to identity in Equation (26), we take as $H^{(2)}_{J(n),2}\left(2^{n-2}\right)=J_{(n-2,n+4)}$. Thus, it is $H^{(2)}_{J(n),2}\left(2^{n-2}\right)=J_{(n-2,6)}=J_6$, $n\equiv 2\,(mod\ 6)$, or $H^{(2)}_{J(n),2}\left(2^{n-2}\right)=J_3$, $n\equiv 5\,(mod\ 6)$. Otherwise it is observed that $H^{(2)}_{J(n),2}\left(2^{n-2}\right)=J_{(n-2,6)}$ is J_2, $n\equiv 0,4\,(mod\ 6)$; or J_1, $n\equiv 1,3\,(mod\ 6)$.

It is known that

$$\left(J_n,J_{n+3}\right)=3\ \textit{if only if}\ n\equiv 0\,(mod\ 3)\,,\ \textit{otherwise}\ \left(J_n,J_{n+3}\right)=1.$$

Now, we establish the sequences $\left\{G^{(2)}_{J(n),3}\left(2^{n-1}\right)\right\}$ and $\left\{H^{(2)}_{J(n),3}\left(2^{n-2}\right)\right\}$ with the following numbers.

Theorem 7. Let $G^{(2)}_{J(n),3}\left(2^{n-1}\right)$ and $H^{(2)}_{J(n),3}\left(2^{n-2}\right)$ be the n^{th} *3*-successive altered Jacobsthal gcd numbers. They are valid:

$$G^{(2)}_{J(n),3}\left(2^{n-1}\right)=\begin{cases} J_5J_3, & n\equiv 1,11\,(mod\ 15)\\ J_5, & n\equiv 6\,(mod\ 15)\\ 1, & n\equiv 0,3,9,12\,(mod\ 15)\\ J_3, & otherwise\end{cases}$$

$$H^{(2)}_{J(n),3}\left(2^{n-2}\right)=\begin{cases} J_7J_3, & n\equiv 2,16\,(mod\ 21)\\ J_7, & n\equiv 9\,(mod\ 21)\\ 1, & n\equiv 0,3,6,12,15,18\,(mod\ 21)\\ J_3, & otherwise\end{cases}$$

proof. If we rewrite $G^{(2)}_{J(n),3}\left(2^{n-1}\right)=\left(J_{n+1}J_{n-1},J_{n+4}J_{n+2}\right)$ according to Equation (16). Then by using identity in Equation (25), since $\left(J_{n+1},J_{n-1}\right)=\left(J_{n+4},J_{n+2}\right)=1$, we get

$$G^{(2)}_{J(n),3}\left(2^{n-1}\right)=\left(J_{n-1},J_{n+2}\right)\left(J_{n-1},J_{n+4}\right)\left(J_{n+1},J_{n+4}\right).$$

If each multipliers are analyzed separately: then they are
$\left(J_{n-1},J_{n+4}\right)=J_{(n-1,5)}=J_5$, $n\equiv 1\,(mod\ 5)$; otherwise, $\left(J_{n-1},J_{n+4}\right)=J_1$,
$\left(J_{n-1},J_{n+2}\right)=J_{(n-1,3)}=J_3$, $n\equiv 1\,(mod\ 3)$; otherwise, $\left(J_{n-1},J_{n+2}\right)=J_1$,
$\left(J_{n+1},J_{n+4}\right)=J_{(n+1,3)}=J_3$, $n\equiv 2\,(mod\ 3)$; otherwise, $\left(J_{n+1},J_{n+4}\right)=J_1$.

Thus, we obtain the following results
$G^{(2)}_{J(n),3}\left(2^{n-1}\right)=J_5J_3$, $n\equiv 1\,(mod\ 5)$ and $n\equiv 1\,(mod\ 3)$,
$G^{(2)}_{J(n),3}\left(2^{n-1}\right)=J_5J_3$, $n\equiv 1\,(mod\ 5)$ and $n\equiv 2\,(mod\ 3)$.

Also, the desired results are obtained using the Chinese remainder theorem in all case.

According to Equation (17), if we consider as $H^{(2)}_{J(n),3}\left(2^{n-2}\right)=\left(J_{n+2}J_{n-2},J_{n+1}J_{n+5}\right)$, then according to $\left(J_{n+2},J_{n+1}\right)=1$, for all n and $\left(J_{n-2},J_{n+5}\right)=J_7$, $n\equiv 2\,(mod\ 7)$, by using identity in Equation (27), we get

$$H^{(2)}_{J(n),3}\left(2^{n-2}\right)=\frac{\left(J_7J_{n+2},J_{n+5}\right)\left(J_{n-2},J_7J_{n+1}\right)}{J_7}. \quad (30)$$

Otherwise, they are valid $\left(J_{n+2},J_{n+1}\right)=\left(J_{n-2},J_{n+5}\right)=1$. By using identity in Equation (26), we achieve

$$H^{(2)}_{J(n),3}\left(2^{n-2}\right)=\left(J_{n+2},J_{n+5}\right)\left(J_{n-2},J_{n+1}\right). \quad (31)$$

For the case in Equation (26), let be $\left(J_7J_{n+2},J_{n+5}\right)=d_1$ and $\left(J_{n-2},J_7J_{n+1}\right)=d_2$, $n\equiv 2\,(mod\ 7)$. It is seen that $\left(J_7,J_{n+2}\right)=\left(J_7,J_{n+1}\right)=1$, $n\equiv 2\,(mod\ 7)$. Therefore, we write the identity in Equation (30) as

$$H^{(2)}_{J(n),3}\left(2^{n-2}\right)=\frac{\left(J_{n+2},J_{n+5}\right)\left(J_7,J_{n+5}\right)\left(J_{n-2},J_{n+1}\right)\left(J_{n-2},J_7\right)}{J_7}.$$

Since $\left(J_7,J_{n+5}\right)=\left(J_{n-2},J_7\right)=J_7$, $n\equiv 2\,(mod\ 7)$, we get $H^{(2)}_{J(n),3}\left(2^{n-2}\right)=J_7\left(J_{n+2},J_{n+5}\right)\left(J_{n-2},J_{n+1}\right)$. Also, if the possible cases in Equations (30-31) are analyzed with
$\left(J_{n+2},J_{n+5}\right)=J_{(n+2,3)}=J_3$, $n\equiv 1\,(mod\ 3)$; otherwise, $\left(J_{n+2},J_{n+5}\right)=J_1$,
$\left(J_{n-2},J_{n+1}\right)=J_{(n-2,3)}=J_3$, $n\equiv 2\,(mod\ 3)$; otherwise, $\left(J_{n-2},J_{n+1}\right)=J_1$.

Then, all values of the sequence $H^{(2)}_{J(n),3}\left(2^{n-2}\right)$ are found by the Chinese remainder theorem for all possible evaluations.

It is known in Equation (32) that

$$\left(J_n,J_{n+4}\right)=5\ \textit{if only if}\ n\equiv 0\,(mod\ 4)\,,\ \textit{otherwise}\ \left(J_n,J_{n+4}\right)=1. \quad (32)$$

We observe that the sequence $\left\{G^{(2)}_{J(n),4}\left(2^{n-1}\right)\right\}$, for $n>2$, exhibits periodic behavior. On the other hand, the sequence $\left\{H^{(2)}_{J(n),4}\left(2^{n-2}\right)\right\}$, for $n\geqslant 2$, assumes values that follow a Jacobsthal sequence.

Theorem 8. Let $G^{(2)}_{J(n),4}\left(2^{n-1}\right)$ and $H^{(2)}_{J(n),4}\left(2^{n-2}\right)$ be n^{th} 4-successive altered Jacobsthal gcd numbers, they are valid:

$$G^{(2)}_{J(n),4}\left(2^{n-1}\right)=\begin{cases} J_6J_4, & n\equiv 1,7\,(mod\ 12)\\ J_4, & n\equiv 3,5,6,10\,(mod\ 12)\\ J_3, & n\equiv 4,8\,(mod\ 12)\\ 1, & otherwise\end{cases},$$

$$H^{(2)}_{J(n),4}\left(2^{n-2}\right)=\begin{cases} J_8J_{n+2}, & n\equiv 2\,(mod\ 8)\\ J_4J_{n+2}, & n\equiv 6\,(mod\ 8)\\ J_{n+2}, & otherwise\end{cases}$$

proof. If we rewrite $G^{(2)}_{J(n),4}\left(2^{n-1}\right)=\left(J_{n+1}J_{n-1},\ J_{n+5}J_{n+3}\right)$ according to Equation (16), then since $\left(J_{n+1},\ J_{n-1}\right)=\left(J_{n+5},\ J_{n+3}\right)=1$, by using identity in Equation (25) for $J_{(n+1,n+3)}=1$, we get

$$G^{(2)}_{J(n),4}\left(2^{n-1}\right)=J_{(n+1,n+5)}J_{(n-1,n+5)}J_{(n-1,n+3)}.$$

If all multipliers are analyzed separately: from Equation (32), it is seen that

$J_{(n+1,n+5)}=J_{(n+1,4)}=J_4,\ n\equiv 3\ (mod\ 4)$, otherwise $J_{(n+1,4)}=1$,

$J_{(n-1,n+3)}=J_{(n-1,4)}=J_4,\ n\equiv 1\ (mod\ 4)$, otherwise $\left(J_{n-1},\ J_{n+3}\right)=1$.

Also, we have $J_{(n-1,n+5)}=J_{(n-1,6)}=J_6,\ n\equiv 1\ (mod\ 6)$; or $J_{(n-1,6)}=J_3,\ n\equiv 4\ (mod\ 6)$; otherwise, it is $J_{(n-1,6)}=1$.

If we consider $n\equiv 1\ (mod\ 6)$ and $n\equiv 1\ (mod\ 4)$ or $n\equiv 1\ (mod\ 6)$ and $n\equiv 3\ (mod\ 4)$, then we can obtain the following result by applying the Chinese remainder theorem:

$$G^{(2)}_{J(n),4}\left(2^{n-1}\right)=J_6J_4,\quad n\equiv 1,7\,(mod\ 12).$$

Similarly, for all possible evaluations, the desired results are obtained.

By using Equation (17), we get $H^{(2)}_{J(n),4}\left(2^{n-2}\right)=J_{n+2}\,J_{(n-2,n+6)}$. Since $J_{(n-2,8)}=J_8,\ n\equiv 2\ (mod\ 8)$, or $J_{(n-2,8)}=J_4,\ \ n\equiv 6\ (mod\ 8)$ or $J_{(n-2,8)}=1$ in the other cases.

The results presented in Theorems 5-8 demonstrate that the lengths of the periods for these numbers can be determined for any value of m, given by

$$m=lcm[r,r-2t,r+2t],\ t\in\{1,2\}\text{ and }1\leqslant r\leqslant 4,\ r-2t\neq 0,$$

where $lcm(a,b,c)$ denotes the least common multiple of three integers a,b and c. Furthermore, these results indicate that the sequences $\left\{G^{(2)}_{J(n),2}\left(2^{n-1}\right)\right\}$ and $\left\{H^{(2)}_{J(n),4}\left(2^{n-2}\right)\right\}$ exhibit similarities to Jacobsthal sequences for $r=2t$ and $t\in\{1,2\}$. To verify whether the sequences $\left\{G^{(2)}_{J(n),r}\left(2^{n-1}\right)\right\}$, for $r\neq 2$ and $\left\{H^{(2)}_{J(n),r}\left(2^{n-2}\right)\right\}$, for $r\neq 4$, are bounded by products of Jacobsthal numbers, a program evaluation was conducted. It was observed that these sequences assume values corresponding to the factors of m as indices and exhibit periodic behavior. Specifically, it was found that these sequences for $5\leq r\leq 50$, are both periodic and bounded sequences with respect to different $(mod\ m)$ values, where $m=lcm[r,r-2t,r+2t]$ (as presented in Table 1).

Algorithm 2 (Table 2), employed in this study, facilitates the identification of maximum values in each column of a data matrix and the detection of any repeated values. The verification process was conducted using a computer program generated by Algorithm 1-2. Multiple values were scrutinized using this program, while considering the provided $(t,\ s)$ parameters.

By considering the derived expressions for $a=2^{(n-t)}J_t^2$, $t\in\{3,4\}$, as given in Equations (21-22), we establish the following:

$$G^{(2)}_{J(n)}\left(2^{n-3}9\right)=J_{n+3}J_{n-3},\ n\geq 3, \tag{33}$$

$$H^{(2)}_{J(n)}\left(2^{n-4}25\right)=J_{n+4}J_{n-4},\ n\geq 4. \tag{34}$$

The Jacobsthal numbers listed below are observed in the set of r-successive altered Jacobsthal Gcd sequences, as indicated by Equation (33-34):

$$G^{(2)}_{J(n),6}\left(2^{n-3}9\right)=\begin{cases} J_{12}J_{n+3}, & n\equiv 3\,(mod\ 12)\\ J_6J_{n+3}, & n\equiv 9\,(mod\ 12)\\ J_4J_{n+3}, & n\equiv 7,11\,(mod\ 12)\\ J_3J_{n+3}, & n\equiv 0,6\,(mod\ 12)\\ J_{n+3}, & otherwise\end{cases},$$

$$H^{(2)}_{J(n),8}\left(2^{n-4}25\right)=\begin{cases} J_{16}J_{n+4}, & n\equiv 4\,(mod\ 16)\\ J_8J_{n+4}, & n\equiv 12\,(mod\ 16)\\ J_4J_{n+4}, & n\equiv 0,8\,(mod\ 16)\\ J_{n+4}, & otherwise\end{cases}.$$

The proofs of these values are not provided here for the sake of brevity, as they follow a similar approach to the proofs of other values.

Theorem 8. Let $2t=r$. It is observed that the r-successive altered Jacobsthal gcd sequences $G^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right)$ and $H^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right)$ exhibit similarities to any Jacobsthal sequences:

$$J_xJ_{n+t}\,(mod\ 4t)=\begin{cases} G^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right), & if\ t\ is\ odd\\ & n,r,t,x\in Z^{+}\\ H^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right), & otherwise\end{cases}$$

where $J_x,\ x\in\{1,2,4,d_i,t,2t,4t\}$, $d_i|\,t,\ i=1,2,\ldots,h,\ d_i\notin\{1,2,4,t,2t,4t\}$ is given as

$$J_x=\begin{cases} J_{4t}, & n\equiv t\,(mod\ 4t)\\ J_{2t}, & n\equiv 3t\,(mod\ 4t)\\ J_t, & n\equiv 0,2t\,(mod\ 4t)\\ J_{d_i}, & n\equiv t+f_id_i\,(mod\ 4t)\\ J_4, & n\equiv t+4t_j\,(mod\ 4t)\\ 1, & otherwise\end{cases},\quad t\neq 1,2,4$$

for $f_i=1,2,\ldots,\frac{t}{d_i}-1$ and $t_j=1,2,3\ldots,t-1$, for $(2,t)=1$ or $t_j=1,3,5,\ldots,t-1$, for $(2,t)\neq 1$.

Table 1. Definitions of Sequences in Algorithm 1

Step	Description
1	Begin
2	Create a list named "n" containing numbers from 0 to 1000
3	Set the first two elements of the list "J" ($J[0] = 0$, $J[1] = 1$)
4	Calculate the values of "G" and "H" using the given formulas with a for loop iterating through "J"
5	Def calculate gcd, which calculates the greatest common divisors (gcd) for different values of "r" for the lists "G" and "H"
6	Create a loop for "r" from 1 to 50
7	Create a loop from "$1000 - r$" to 1
8	For each "r", calculate the gcd for $G[i]$ and $G[i+r]$ and save it in the list "$gcdG[r]$"
9	For each "r", calculate the gcd for $H[i]$ and $H[i+r]$ and save it in the list "$gcdH[r]$"
10	Combine all arrays and the calculated gcd values
11	Create a DataFrame and save it as a CSV file
12	End

Table 2. Identification of Repeated Values and Their Frequencies in Algorithm 2

Step	Description
1	Begin
2	Create a matrix named "repeated" and set its dimensions to 1000×80
3	For start a loop "i" from t to 25
4	For each column "r", from 1 to 40, apply the following steps
5	Assign the i^{th} column of the data matrix to a variable named "$data''$: $data = data(:, i)$
6	Find the maximum value (m) in the "$data''$: $max(data)$
7	Find the positions of "m" in the "$data''$: $g = find(data == m, 4)$
8	Check if the positions of the maximum values exceed the size of the array
9	Calculate the length between $g(2)$ and $g(1)$ ($length$) : $length = g(2) - g(1)$
10	Create arrays $x_1$and x_2 and compare them
11	If x_1 and x_2 are equal, save the values of x_1 in the "repeated"
12	Savex_1 and its frequency
13	If x_1 and x_2 are not equal, check position $g(3)$ and create arrays x_1 and x_3
14	If x_1 and x_3 are equal, save the values of x_1 in the "repeated" matrix and frequency
15	End

Proof. Since we have a multiplication expression for the numbers $G^{(2)}_{J(n)}\left(2^{n-t}J_t^2\right)$ and $H^{(2)}_{J(n)}\left(2^{n-t}J_t^2\right)$ in Equations (21-22), r-successive altered Jacobsthal gcd sequences are rewritten as follows:

$$(J_{n+t}J_{n-t},\ J_{n+t+r}J_{n-t+r}) = \begin{cases} G^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right), & if\ t\ is\ odd \\ H^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right), & otherwise \end{cases}.$$

Considering all possible situations given in Equations (25-27), a complete evaluation is made. However, since the operations are similar, let us shorten some repetitions. Firstly, assume that the value of $J_{(n-t,2t)} = J_{(n-t+r,2t)} = 1$ in Equation (25) is satisfied. We write

$$J_{(n+t,n+t+r)}J_{(n+t,n-t+r)}J_{(n-t,n+t+r)}J_{(n-t,n-t+r)} = \begin{cases} G^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right), & \text{if } t \text{ is odd} \\ H^{(2)}_{J(n),r}\left(2^{n-t}J_t^2\right), & \text{otherwise} \end{cases}$$

Since $J_{(n+t,r)} = J_{(n-t,r)} = 1$ for $2t = r$, we obtained as

$$J_{(n+t,r)}J_{(n+t,-2t+r)}J_{(n-t,2t+r)}J_{(n-t,r)} = J_xJ_{n+t}$$

where $J_x = J_{(n-t,4t)}$ and $J_{(n+t,0)} = J_{n+t}$. It is seen that J_x has the solution for $m = 4t$.

If $n - t = 4tk_1$, then $n \equiv t\ (mod\ 4t)$, it is $J_x = J_{4t}$.

For $n - t = 2tk_2$, $(k_2, 4) = 1$, then $n \equiv 3t\ (mod\ 4t)$, it is $J_x = J_{2t}$.

For $n - t = tk_3$, $(k_3, 4) = 1$, then $n \equiv 0, 2t\ (mod\ 4t)$, it is $J_x = J_t$.

If t is not prime, we consider that it has h positive divisors; $d_i\,|t$, $i = 1, 2, \ldots, h$.

Let be $n - t = d_ik_i$, then $n \equiv t + f_id_i\ (mod\ 4t)$, $(f_i, t) = 1$, $f_i = 1, 2, \ldots, \frac{t}{d_i} - 1$, it is $J_x = J_{d_i}$.

For $n - t = 4$, $(t, 2) = 1$, then $n \equiv t + 4t_j\ (mod\ 4t)$, $t_j = 1, 2, \ldots, t - 1$, it is $J_x = J_4$.

For $n - t = 4$, $(t, 2) \neq 1$, then $n \equiv t + 4t_j\ (mod\ 4t)$, $t_j = 1, 3, \ldots, t - 1$, it is $J_x = J_4$.

Otherwise, $J_x = J_2 = J_1$.

Secondly, assume that the value of $J_{(n+t,n+t+r)} = J_{(n-t,n-t+r)} = 1$ in Equation (26) is satisfied, we get $J_{(n+t,r)} = J_{(n-t,r)} = 1$. Thus, it is seen that $J_{(n+t,0)} J_{(n-t,4t)} = J_x J_{n+t}$.

Thirdly, assume that the value of $J_{(n+t,r)} = x$, $J_{(n-t,r)} = y$, $r = 2t$ in Equation (27) is satisfied. Thus, $J_{(n-t,2t)} = x$, $J_{(n+t,2t)} = y$. We rewrite

$$\frac{(x,y)\ (xJ_{n-t}, yJ_{n+3t})}{xy} J_{n+t} = \begin{cases} G^{(2)}_{J(n),r}\left(2^{n-t} J_t^2\right), & if\ t\ is\ odd \\ H^{(2)}_{J(n),r}\left(2^{n-t} J_t^2\right), & otherwise \end{cases}$$

A calculation program is implemented to verify the hypothesis for values up to $t = 25$ and $r = 50$, considering different x and y values. It is observed that the hypothesis held true in all cases.

For the cases of $G^{(2)}_{J(n),r}\left(2^{n-3}9\right)$, $r \neq 6$ and $H^{(2)}_{J(n),r}(2^{n-4}25)$, $r \neq 8$, it is observed that the sets of r-numbers in the r-successive altered Jacobsthal gcd sequences, modulo m, exhibited periodic behavior based on the expression $m = lcm\left[r, 2t - r, 2t + r\right]$, $r \neq 2t$. For instance, the lengths of the periods for the numbers $G^{(2)}_{J(n),r}\left(2^{n-3}9\right)$ and $H^{(2)}_{J(n),1}\left(2^{n-4}25\right)$, $r = 1 - 4$ are determined by

$$G^{(2)}_{J(n),1}\left(2^{n-3}9\right) = \{J_7 J_5, J_7, J_5, 1\}\ (mod\ 35)$$

$$G^{(2)}_{J(n),2}\left(2^{n-3}9\right) = \begin{cases} J_8, & n \equiv 3\ (mod\ 8) \\ J_4, & n \equiv 1, 5, 7\ (mod\ 8) \\ 1, & otherwise \end{cases}$$

$$G^{(2)}_{J(n),3}\left(2^{n-3}9\right) = \begin{cases} J_3 J_9, & n \equiv 3\ (mod\ 9) \\ J_3^2, & n \equiv 0, 6\ (mod\ 9) \\ 1, & otherwise \end{cases}$$

$$G^{(2)}_{J(n),4}\left(2^{n-3}9\right) = \begin{cases} J_4 J_{10}, & n \equiv 3\ (mod\ 10) \\ J_5, & n \equiv 8\ (mod\ 10) \\ J_4, & n \equiv 1, 5, 7, 9, (mod\ 10) \\ 1, & otherwise \end{cases}$$

$$H^{(2)}_{J(n),1}\left(2^{n-4}25\right) = \left\{J_9 J_7, J_9, J_7, J_3^2, J_3, 1\right\}\ (mod\ 63)$$

$$H^{(2)}_{J(n),2}\left(2^{n-4}25\right) = \{J_{10} J_6, J_{10}, J_5 J_3, J_6, J_5, J_3, 1\}\ (mod\ 30)$$

$$H^{(2)}_{J(n),3}\left(2^{n-4}25\right) = \{J_{15} J_{11}, J_{15}, J_{11}, J_5 J_{11}, J_3 J_{11}, J_3 J_5, J_5, J_3, 1\}\ (mod\ 165)$$

$$H^{(2)}_{J(n),4}\left(2^{n-4}25\right) = \left\{J_{12} J_5, J_6, J_4^2, J_3, 1\right\}\ (mod\ 12).$$

Furthermore, by utilizing the implemented calculation program, examples of the sequences $G^{(2)}_{J(n),r}\left(2^{n-3}9\right)$, $r \neq 6$ and $H^{(2)}_{J(n),r}\left(2^{n-4}25\right)$, $r \neq 8$, were evaluated for r values ranging from 1 to 50. It was observed that all of these sequences are both bounded and periodic.

The lengths of the periods and the corresponding period values modulo m for both the Jacobsthal sequences and the Jacobsthal product-valued sequences are of particular interest. It is worth noting that a formula, $m = lcm\left[r, 2t - r, 2t + r\right]$, $(2t \neq r)$, can be employed to determine the lengths of the periods modulo m.

3. RESULT AND DISCUSSION

In this study, we introduced altered Jacobsthal numbers squared and explored their properties, focusing on their relationships with Jacobsthal multiplication patterns and gcd sequences. The results show that the sequences $G^{(2)}_{J(n)}(a)$ and $H^{(2)}_{J(n)}(a)$ provide valuable insights into the divisibility and periodicity of these numbers. While the connections to Jacobsthal numbers are evident, the exact nature of their periodic behaviors, especially for generalized forms, warrants further investigation (Koken, 2019).

The study's findings suggest that these altered sequences could serve as a foundation for exploring broader classes of number sequences with similar properties. The periodicity and boundedness of certain sequences, as well as the existence of specific patterns in their generalizations, indicate that there may be deeper algebraic or number-theoretic structures at play, which could be explored in future work.

Several avenues for further research emerge from this study. First, the study of the gcd sequences for higher values of r and t could provide more comprehensive results, especially with larger computational ranges. Extending the algorithms to explore sequences for even greater values of r and t may reveal new patterns or unexpected properties. Additionally, exploring the practical applications of these findings, such as in cryptography or coding theory, could help in understanding the real-world implications of the mathematical properties uncovered in this research.

Further theoretical work may also be needed to explore the deeper relationships between these altered sequences and other well-known sequences, such as Fibonacci or Lucas numbers (Koshy, 2019). Investigating how these sequences fit within the broader context of integer sequences and their applications to combinatorics or number theory could prove fruitful.

4. CONCLUSIONS

This study introduces and examines the properties of altered Jacobsthal numbers squared and their associated gcd sequences. By defining the sequences $G^{(2)}_{J(n)}(a)$ and $H^{(2)}_{J(n)}(a)$, their connection to Jacobsthal multiplication patterns is established. Furthermore, it was demonstrated that their generalizations exhibit the same unique Jacobsthal multiplication pattern as

$$J_{n+t} J_{n-t} = \begin{cases} G^{(2)}_{J(n)}\left(2^{n-t} J_t^2\right), & if\ t\ is\ odd \\ H^{(2)}_{J(n)}\left(2^{n-t} J_t^2\right), & otherwise \end{cases}$$

The generalizations and periodicity of these sequences are analyzed, revealing their relationship to traditional Jacobsthal subsequences and bounded periodic behaviors. Additionally, closed-form expressions, Binet-like formulas, and computational algorithms are presented, extending the analysis to broader parameter ranges. These findings provides insights into the properties of altered Jacobsthal numbers squared and their gcd sequences, offering a deeper understanding of their mathematical behavior and offering potential applications in various fields.

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