Science and Technology Indonesia
e-ISSN:2580-4391 p-ISSN:2580-4405
Vol. 9, No. 4, October 2024

Research Paper

Probabilistic Multi-Item Inventory Model for Chemicals in Regional Drinking Water Company

Oki Dwipurwani[1,2], Fitri Maya Puspita[2*], Siti Suzlin Supadi[3], Evi Yuliza[2]

[1]Mathematics and Natural Sciences Doctoral Study, Sriwijaya University, Ogan Ilir, 30662, Indonesia
[2]Mathematics Department, Faculty of Mathematics and Natural Sciences, Sriwijaya University, Ogan Ilir, 30662, Indonesia
[3]Institut of Mathematical Sciences, University of Malaya, Kuala Lumpur, 50603, Malaysia
*Corresponding author: fitrimayapuspita@unsri.ac.id

Abstract
Inventory management of chemicals in The Regional Drinking Water Company (Perusahaan Daerah Air Minum, PDAM) is known to play an essential role in ensuring the smooth production of clean water and preventing shortages of chemicals that can affect production. At present, 2 primary models for multi-item inventory replenishment are under consideration by PDAM, namely individual and joint replenishment of items. Therefore, this study aims to evaluate the efficiency and effectiveness of individual and joint replenishment models based on uncertain demand with specific probability distribution. The results showed that a probabilistic inventory model (Q, r) with individual replenishment for chemicals in PDAM was recommended.

Keywords
PDAM, Inventory Replenishment, Probabilistic Inventory Model

Received: 23 June 2024, Accepted: 29 September 2024
https://doi.org/10.26554/sti.2024.9.4.1024-1032

1. INTRODUCTION

Regional Drinking Water Company (PDAM) is responsible for the distribution of clean water to the community. Several reports showed that the production of clean water comprised the use of various chemicals, including liquid aluminum sulfate, chlorine gas, lime, and calcium hypochlorite. Liquid aluminum sulfate is typically used as a dirt binder, while lime is often introduced for neutralization, achieving a pH of > 7 for further clarification. In addition, chlorine gas and calcium hypochlorite (CL_2) are applied as disinfectants to eliminate all inherent bacteria in the water, leading to its distribution to all customers through pipelines (Fitria et al., 2015).

The availability of these chemicals is very essential as their scarcity can lead to significant losses for PDAM. These losses include both financial setbacks due to delays in water distribution and a decline in public trust in the company. Consequently, effective inventory management becomes crucial to establishing an optimal chemicals inventory quantity policy.

According to previous studies, effective inventory management is intricately tied to the average demand for chemicals in a planning horizon. This demand is subject to uncertainty, fluctuating at times and following a specific probability distribution. In addition, an inventory model designed for such demand characteristics can take the form of a probabilistic or fuzzy inventory model. Several reports examined probabilistic inventory model specifically tailored to demand with a normal probability distribution, either for simulation or historical data (Gutierrez and Rivera, 2021; Rojano et al., 2020; Sutoni and Taufik, 2019). Inventory modeling also extends to demands characterized by probability distribution other than normal, including Exponential, Gamma, Weibull, and Pareto (Christou et al., 2020; Hansen et al., 2022; Lesmono and Limansyah, 2019, Salas-Navarro et al., 2020; Zhang et al., 2022). In instances where there is no specific distribution or high uncertainty, the fuzzy inventory model presents an effective solution (Das and Islam, 2024; De, 2021; Saha et al., 2023; Susanti et al., 2023).

Apart from uncertain demand, inventory replenishment for multi-item scenarios comprises 2 main methods, namely individual stock and joint replenishment policy. Previous studies showed that joint replenishment became applicable when the vendor was the same.

A probabilistic multi-item inventory model, designed to address demand uncertainty with a specific probability distribution, is a strategic method in inventory management. In the contemporary landscape of business or public service, characterized by pervasive uncertainty, responsive inventory management becomes indispensable. This model accounts for the uncertainty surrounding the demand for multiple products or items. By acknowledging the random nature of demand for each item in stock through the understanding and modeling of a probability distribution, the probabilistic multi-item inventory model aims to facilitate informed decision-making and enhance overall efficiency.

An exemplary development is the joint replenishment inventory model, specifically designed for (R, T) model (Lesmono and Limansyah, 2017; Wang and Chen, 2022). Silitonga et al. (2021) conducted a study comparing (r, Q) and (R, T) models for demand with a normal probability distribution under joint replenishment. The (Q, r) model is also called the Fixed-Order Quantity Inventory Model, and the (R, T) model is also called the Fixed-Order Interval Inventory Model. In addition, joint replenishment policies controlled by sales thresholds, with Poisson demand, have been extensively explored by Wan et al. (2022). Castellano and Santillo (2023) carried out an investigation assuming random demand with a normal probability distribution, controllable lead times, and the implementation of fill rate constraints for each item. Generally, research determines that the probability distribution of demand for each item is the same, but in this paper, the probability distribution of demand is different.

Previous studies, specifically regarding modeling chemicals supplies in PDAM, generally used one type of chemical with demand having a normal probability distribution (Ayu et al., 2022; Dewi et al., 2019). Based on the results, there is no exploration of the inventory model considering demand distribution other than normal and joint replenishment policies in PDAM. The demand for chemical materials by the production department is also uncertain in quantity but follows a specific probability distribution. Therefore, this aims to assess probabilistic inventory models (Q, r) and (R, T) for multi-items, considering both individual and joint replenishment policies mathematically, using historical data from PDAM Tirta Musi Palembang.

2. EXPERIMENTAL SECTION

2.1 Data Source

Secondary data were obtained from PDAM Tirta Musi Palembang, taken from June to January 2024. The data consists of demand data for liquid aluminum sulfate, lime, chlorine gas, and calcium hypochlorite by the production department to the warehouse and procurement department. Other data included cost-related information, including purchase, storage, and shortage costs for each chemical, obtained from the procurement, general, and financial departments.

2.2 Methods

The study method consisted of testing assumptions regarding the probability distribution of chemical material demand and probabilistic multi-item inventory modeling to obtain optimal inventory policies. To test the assumptions regarding the probability distribution of chemical material demand, The Kolmogorov-Smirnov (KS) method was used, with the following statistical tests:

$$KS_{value} = sup_{x_j} |F_n(x_j) - F_0(x_j) \tag{1}$$

Where $F_n(x_j)$ is the cumulative probability of a particular distribution, and $F_0(x_j)$ is the cumulative probability of the empirical data tested. The hypothesis was H_0 = The residuals are of the specified distribution and H_1 = The residuals do not have a specified distribution. Furthermore, H_0 rejection region was $\mathrm{KS}_{value} > \mathrm{KS}_{\alpha,n}$ or $\mathrm{KS}p_{value} < \alpha$ (Otsu and Taniguchi, 2020).

Apart from KS value, distribution suitability testing could use Bayesian Information Criterion (BIC) and Akaike Information Criterion (AIC) values (de Figueiredo et al., 2023), each of which had the following formulation:

$$AIC = -2Log(L) + 2(p + 1) \tag{2}$$

$$BIC = -2Log(L) + (p + 1)log(n) \tag{3}$$

Where L is the estimated likelihood function, p is the number of parameters in the probability density function, and n is the number of observations. The smaller the AIC and BIC values, the better the consistency with the specified probability distribution.

The assumptions for inventory model with individual replenishment and joint replenishment were demand at lead time followed a certain probability distribution, unlimited warehouse capacity, lead time of all chemicals was the same and constant, chemicals shortages were permitted and occurred only at the lead time (L) for (Q, r) individual filling model, at $(T + L)$ for (R, T) individual filling and joint filling models, handling inventory shortages was handled with a back order system, and the order quantity was fixed for each order period. The definitions of the variables and parameters used are presented in Table 1.

To obtain the optimum policy in inventory model, an exact method was used through the Hadley-Within Algorithm, namely by minimizing the total cost (Tc) of Q_j, r_j, T_j, and T_{joint}, that fulfilled the conditions $\frac{\partial T_c(Q_j,r_j)}{\partial Q_j} = 0$, $\frac{\partial T_c(Q_j,r_j)}{\partial r_j} = 0$, $\frac{\partial T_c(R_j,T_j)}{\partial T_j} = 0$ and $\frac{\partial T(joint)}{\partial T_{joint}} = 0$ (Bahagia, 2006; Christou et al., 2020; Dolgan et al., 2020; Silver et al., 2017). The total cost was the sum of purchasing cost (Pc), ordering cost (Oc), holding cost (Stc), and shortage cost (Shc).

Inventory model built included individual replenishment (Q_j, r_j) (Model 1), individual replenishment (R_j, T_j) (Model 2), and the joint replenishment models (Model 3).

The total cost formulation for Model 1 was:

$$T_{c_j}(Q_j, r_j) = D_j.p_j + \left(A_1 + \frac{A_2 D_j}{Q_j}\right) + h_j\left(\frac{1}{2}Q_j + r_j - (D_j L_j)\right) + \frac{C_{u_j} D_j}{Q_j} \int_r^\infty (x_j - r_j) f(x_j) dx_j \tag{4}$$

Table 1. The Definitions Variables and Parameters

Variables and Parameters	The Definitions of Variables and Parameters
η_j	Service level of chemicals j
N_j	Expected amount of inventory shortage each cycle (unfulfilled demand), for chemicals j
D_{L_j}	Demand expectations during the lead time period for chemicals j
α_j	Percentage of unfulfilled requests for chemicals j
r_j	The amount of inventory at the time the order was placed (reorder point) for chemicals j
X_j	The random variable of demand for chemicals j
$f(x_j)$	Demand opportunity density function at lead-time for chemicals j
j	Chemicals index
D_j	Demand expectations over the planning horizon for chemicals j
Lj	lead time for chemicals j
Q_j	The order lot size for each order for chemicals j
p_j	The purchase cost of goods for chemicals j
A_2	Cost of contracting
A_1	Order cost
h_j	Holding cost per unit of chemicals j
C_{u_j}	The unit cost of inventory shortage per unit of chemicals j
Tc_j	Total cost of chemicals j
Pc_j	Cost of purchasing for chemicals j
Oc_j	Ordering costs of chemicals j
Stc_j	Storage costs of chemicals j
Shc_j	Shortage costs of chemicals j
ss_j	Safety stock of chemicals j
T	Joint charging time interval
T_j	Refill time interval of chemicals j
R_j	Maximum supply of chemicals j
Z_j	Random variable demand time $(T + L)$ chemicals j
$\theta_j, \eta_j, \lambda_j, \beta_j, \gamma_j$	Probability distribution parameters of chemical j

The total cost formulation for Model 2 was:

$$T_{c_j}(R_j, T_j) = D_j p_j + \left(A_1 + \frac{A_2}{T_j}\right) + h_j\left(R_j - (D_j L_j) + \frac{D_j T_j}{2}\right) + \frac{C_{u_j}}{T_j}\int_{R_j}^{\infty}(z_j - R_j)f(z_j)dz_j \tag{5}$$

The total cost formulation for Model 3 was:

$$T_c(Joint) = \left(A_1 + \frac{A_2}{T_{joint}}\right) + \sum_{j=1}^{4}\left\{(D_j p_j + h_j \left(R_j - (D_j L_j) + \frac{D_j T_{joint}}{2}\right) + \frac{C_{u_j}}{T_{joint}} \int_{R_j}^{\infty}(z_j - R_j)f(z_j)dz_j\right\} \tag{6}$$

Where $j = 1, 2, 3, 4$ is the index for chemicals. $j = 1$ is the liquid aluminum sulfate index, $j = 2$ is the chlorine gas index, $j = 3$ is the lime index, and $j = 4$ is the calcium hypochlorite index.

In Model 1, the state $\frac{\partial T_c(Q_j, r_j)}{\partial Q_j} = 0$ gave the order lot size for each chemical j (Q_j) formulated by:

$$Q_j = \sqrt{\frac{2D_j(A_2 + C_{u_j}N_j)}{h_j}} \tag{7}$$

With $N_j = \int_{r_j}^{\infty}(x_j - r_j)f(x_j)dx_j$ as the amount of chemical inventory shortage for j.

From the circumstance $\frac{\partial T_{c_j}(Q_j, r_j)}{\partial r_j} = 0$, the formulation was obtained:

$$\int_{r_j}^{\infty} f(x_j)dx_j = \frac{Q_j h_j}{C_{u_j} D_j} = \alpha_j \tag{8}$$

Equation (8) showed the percentage shortage of chemicals j (α_j), which could be used to determine the reorder point for chemical j (r_j).

In Model 2, the state $\frac{\partial T_c(R_j, T_j)}{\partial T_j} = 0$ yielded a formulation for the individual filling time of chemical j (T_j):

$$T_j = \sqrt{\frac{2(A_2 + C_{u_j}\int_{R_j}^{\infty}(z_j - R_j)f(z_j)dz_j)}{h_j D_j}} \tag{9}$$

Table 2. Suitability of Probability Distribution to Chemicals Demand Data

Index (j)	Type of Chemical	Types of probability Distribution	KSp Value	KS Value	AIC	BIC
1	Liquid Aluminum Sulfate	Normal	0.965	0.160	244.22	244.38
		Gamma	0.968	0.159	246.23	246.47
		Exponential	0.544	0.265	240.39	240.47
		Erlang	0.966	0.159	246.23	246.47
		Weibull	0.947	0.169	246.17	246.41
		Pareto	0.100	0.409	250.90	251.06
2	Chlorine Gas	Gamma	0.739	0.224	202.67	202.92
		Exponential	0.077	0.427	204.41	204.49
		Weibull	0.539	0.266	204.45	204.68
		Normal	0.419	0.292	210.88	211.04
3	Lime Powder	Gamma	0.746	0.222	180.11	180.35
		Exponential	0.544	0.265	240.39	240.47
		Erlang	0.438	0.288	211.81	211.05
		Weibull	0.517	0.271	214.67	214.91
4	Calcium Hypochlorite	Normal	0.266	0.334	163.59	163.75
		Gamma	0.266	0.335	165.59	165.83
		Weibull	0.282	0.329	161.49	161.73

Table 3. Demand and Cost Data in Inventory Model

Index (j)	Chemicals Material	Demand (kg)	Fees are in IDR: Booking contract/ year(A_1)	Ordering cost (A_2)	Purchase cost/kg (p) with 11% VAT	Holding costs (h) 10% purchase fee	Shortage cost (Cu)
1	Aluminum	7,121,888.49	51,000,000	5,000	1,736.17	173.62	1,822.97
2	Chlorine Gas	124,626.94	51,000,000	5,000	21,534.59	2,153.46	22,611.32
3	Lime	638,885.27	51,000,000	5,000	3,520.00	352.00	3,696.00
4	Calcium Hypochlorite	6,112.99	51,000,000	5,000	40,300.00	4,030.00	42,315.00

In Model 3, the state $\frac{\partial T_c(Joint)}{\partial T} = 0$ provided a formulation of the filling time for chemicals T_{Joint}:

$$T_{Joint} = \sqrt{\frac{2(A_2^* + \sum_{j=1}^{n} C_{u_j} \int_{R_j}^{\infty} (z_j - R_j) f(z_j) dz_j)}{\sum_{j=1}^{n} h_j D_j}} \quad (10)$$

With an ordering lot size for each chemical of $Q_j = T_{Joint} \times D_j$, and n is the number of items.

From the necessary conditions of $\frac{\partial T_c(R,T)}{\partial R_j} = 0$ and $\frac{\partial T_c(Joint)}{\partial R_j} = 0$, a formulation for the percentage of chemical j (α_j) deficiency was obtained, as in Equation (8), namely: $\int_{R_j}^{\infty} f(z_j) dz_j = \frac{T_j h_j}{C_{u_j}} = \alpha_j$ for Model 2, and $\int_{R_j}^{\infty} f(z_j) dz_j = \frac{T_{Joint} \times h_j}{C_{u_j}} = \alpha_j$ for Model 3. In addition, the maximum inventory value for all chemicals j (R_j) was also obtained.

The optimum solution for all chemicals j in Model 1 was carried out by iteratively calculating the values of α_j and r_j in Equation (8), until a relatively constant value of r_j was obtained. Models 2 and 3 were also obtained through iteration, with the T_j value calculated using Equation (9) or T_{Joint} using Equation (10). Subsequently, the total cost $T_{c_j}(R_j, T_j)$ was calculated using Equation (5), and the total inventory cost $T_c(Joint)$ with Equation (6). Iteration was then carried out with: $T_c(R_j, T_j) = T_c(R_j, T_j) \pm \Delta T_c(R_j, T_j)$ or $T_c(Joint) = T_c(Joint) \pm \Delta T_c(Joint)$, until it gave the minimum total cost of $T_c(R_j, T_j)$ or $T_c(Joint)$. All calculations in this study were performed using the Math Library in Python software.

3. RESULTS AND DISCUSSION

3.1 Study Data

The production of clean drinking water at PDAM was carried out in the water production department, through various stages of the coagulation, flocculation, sedimentation, aeration, and filtration processes. In the coagulation process, liquid aluminum sulfate was used as a coagulant to break down solid particles dissolved in water. During the flocculation process, the particles combined and settled to facilitate the removal. Aeration increased the oxygen content in the water, while filtration was carried out to remove the particles using a filtering medium. To raise the pH of the clean water, lime chemicals were used. Subsequently, chlorine gas and calcium hypochlorite chemicals were used for the disinfection of disease-causing microorganisms, and removal of color, odor, and taste, ensuring that the

Table 4. Optimum Policy Results for Model 1 (i = 1)

Variable	Aluminium (j=1) (Exponential)	Chlorine Gas (j=2) (Gamma)	Lime (j=3) (Gamma)	Calcium Hypochlorite (j=4) (Weibull)	Total
$Tc(Q_j, r_j)$ (IDR)	12,388,971,222.61	2,703,564,340.64	2,263,819,152.34	260,425,049.58	17,616,779,765.17
Iteration to	9	7	5	5	
r_{1j} (kg)	75,955.79	2,777.85	5,096.75	197.63	
Q_{1j} (kg)	26,653.19	1,129.34	4,441.56	161.96	
N_{1j} (kg)	2.01	0.27	0.12	0.086	
α_{1j} (%)	0.00036	0.00086	0.00066	0.00252	
ss_{1j} (kg)	39,078.07	2,132.52	1,788.55	165.97	
η_{1j} (%)	99.99	99.95	99.99	99.72	
Stc_{1j} (IDR)	9,098,341.49	5,808,288.02	1,411,286.41	995,208.92	17,313,124.84
Pc_{1j} (IDR)	12,364,809,157.05	2,683,790,055.85	2,248,876,150.40	246,353,497.00	17,543,828,860.30
Oc_{1j} (IDR)	14,086,028.93	13,301,767.51	13,469,211.79	12,938,722.44	53.795.730,67
Shc_{1j} (IDR)	977,695.14	664,229.26	62,503.74	137,621.22	1,842,049.36

Table 5. Optimum Policy Results for Model 3 (i = 2)

Variable	Aluminium (j=1) (Exponential)	Chlorine Gas (j=2) (Gamma)	Lime (j=3) (Gamma)	Calcium Hypochlorite (j=4) (Weibull)	Total
$Tc(R_j, T_j)$ (IDR)	12,395,155,600.57	2,708,510,421.51	2,265,476,260.29	262,047,057.19	17,753,491,691.33
Iteration to	2	5	4	15	
T_{2j} (years)	0.0018	0.0021	0.0036	0.0061	
R_{2j} (kg)	108,817.98	4,557.09	7,626.90	541.45	
$Q_{2j} = D_{2j}T_{2j}$ (kg)	13,389.45	268.78	1,453.67	37.58	
N_{2j} (kg)	1.37	0.087	0.053	2.04	
α_{2j} (%)	0.00017	0.00020	0.00021	0.00059	
ss_{2j} (kg)	58,550.80	3,644.98	2,865.03	472.22	
η_{2j} (%)	99.99	99.99	99.99	99.99	
Stc_{2j} (IDR)	13,652,371.25	8,711,075.06	1,776,029.29	2,130,204.12	26,269,679.72
Pc_{2j} (IDR)	12,364,809,157.04	2,683,790,055.85	2,248,876,150.40	246,353,497.00	17,543,828,860.29
Oc_{2j} (IDR)	15,409,513.12	15,058,727.77	14,947,489.20	135,633,342.01	181,049,072.10
Shc_{2j} (IDR)	1,332,674.86	925,663.46	85,726.83	14.07	2,344,079.22

water remained sterile.

The request for chemicals by the production department was reported to the warehouse, which was then fulfilled through purchases by the procurement department. The data obtained from PDAM were chemicals demand data, which were in the form of monthly and annual time series datasets from January 2016 to December 2023. Other data were in the form of cost data included in the model, such as purchasing, ordering, holding, and shortage cost.

3.2 Testing the Distribution of Demand for Chemicals

Testing of the probability distribution assumption on chemicals materials demand data for one planning horizon (one year) was carried out using KS test Equation (1). The test results that met the distribution of each chemical are presented in Table 2. The smallest AIC and BIC values and KS test met the null hypothesis or the KSp value of 0.05, which was used in inventory modeling.

3.3 Optimum Policy Results of Inventory Model

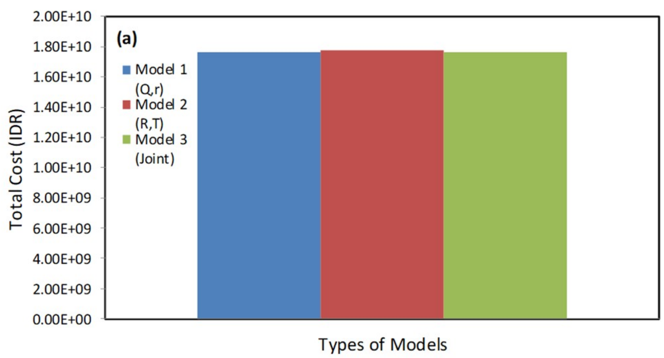
Based on Table 2, for the liquid aluminum sulfate demand, Exponential probability distribution was selected with parameter values Scale (β_1) = 1,087,533.41 and Loc (γ_1) = 6,034,355.09. Meanwhile, Gamma probability distribution was selected for chlorine gas demand with parameter values Shape (λ_1) = 2.35, Scale (θ_2) = 52,987.19. For Lime demand, Gamma probability distribution was selected with parameter values Shape (λ_3) = 46.12, Scale (θ_3) = 13,851.33. Weibull probability distribution was selected for calcium hypochlorite demand with parameter values Shape (β_4) = 0.97 and Scale (η_4) = 6,036.49.

3.4 Probabilistic Inventory Model of Individual Replenishment and Joint Replenishment

Demand data and estimated cost used to implement the three models were presented in Table 3. The results of this study showed that the demand for chemicals was in kilograms (kg). The cost inherent in the model were in rupiah units, consisting of contract cost (A_1) and ordering cost by telephone (A_2),

Table 6. Optimum Policy Results for Model 3 (i = 3)

Variable	Aluminium (j=1) (Exponential)	Chlorine Gas (j=2) (Gamma)	Lime (j=3) (Gamma)	Calcium Hypochlorite (j=4) (Weibull)	Total
Tc(Joint) (IDR)	17,624,355,227.64				17,624,355,227.64
Iteration to	4				
T_3 (years)	0.0017				
R_{3j} (kg)	10,636.11	4,361.15	7,071.05	386.38	
Q_{3j} (kg)	4,708.72	82.39	422.41	4.04	
N_{3j} (kg)	1.18	0.06	0.03	0.00	
α_{3j} (%)	0.00016	0.00016	0.00016	0.00016	
ss_{3j} (kg)	57,655.78	3,508.79	2,701.56	344.57	
η_{3j} (%)	99.99	99.99	99.99	99.99	
Stc_{3j} (IDR)	13,091,016.90	8,224,787.03	1,511,309.87	1,450,019.37	24,277,133.17
Pc_{3j} (IDR)	12,364,809,139.68	2,683,790,055.85	2,248,876,150.40	246,353,497.00	17,543,828,842.93
Oc_{3j} (IDR)	54,009,940.07				54,009,940.07
Shc_{3j} (IDR)	1,291,347.14	863,141.59	77,701.84	7,120.90	2,239,311.47



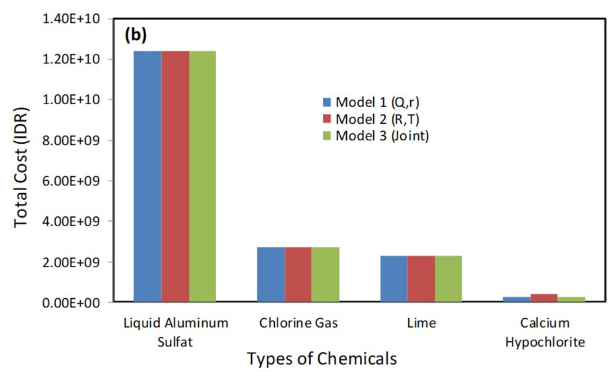


Figure 1. Comparison of Total Inventory Model Cost (a) All Chemicals (b) Each Chemical

purchasing cost (p), holding cost (h), and shortage cost (Cu).

The average value of lead time (L) for all chemicals is 1.89 days or 0.005174 years.

For Model 1, the probability distribution of demand data was Exponential, Gamma, and Weibull. Hence, the development of the lot size formulation for ordering chemicals j (Q_j) in Equation (7), adjusted for the probability density function (Ophokenshi et al., 2019; Sayed and Muhammad, 2022), is as follows:

$$Q_1 = \sqrt{\frac{2D_1(A_2 + C_{u_1}N_1)}{h_1}} \tag{11}$$

$$Q_2 = \sqrt{\frac{2D_2(A_2 + C_{u_2}N_2)}{h_2}} \tag{12}$$

$$Q_3 = \sqrt{\frac{2D_3(A_2 + C_{u_3}N_3)}{h_3}} \tag{13}$$

$$Q_4 = \sqrt{\frac{2D_4(A_2 + C_{u_4}N_4)}{h_4}} \tag{14}$$

The amount of chemical inventory shortage j (N_j) was:

$$N_1 = \int_{r_1}^{\infty} (x_1 - r_1)\frac{1}{\beta_{L_1}} e^{-\frac{x_1 - \gamma_{L_1}}{\beta_{L_1}}} dx_1 \tag{15}$$

$$N_2 = \int_{r_2}^{\infty} (x_2 - r_2)\frac{x_2^{\lambda_{L_2}-1} e^{-x_2/\theta_{L_2}}}{\theta_{L_2}^{\lambda_{L_2}} \Gamma(\lambda_{L_2})} dx_2 \tag{16}$$

$$N_3 = \int_{r_3}^{\infty} (x_3 - r_3)\frac{x_3^{\lambda_{L_3}-1} e^{-x_3/\theta_{L_3}}}{\theta_{L_3}^{\lambda_{L_3}} \Gamma(\lambda_{L_3})} dx_3 \tag{17}$$

$$N_4 = \int_{r_4}^{\infty} (x_4 - r_4)\frac{\beta_{L_4}}{\eta_{L_4}}\left(\frac{x_4}{\eta_{L_4}}\right)^{\beta_{L_4}-1} e^{-\left(\frac{x_4}{\eta_{L_4}}\right)^{\beta_{L_4}}} dx_4 \tag{18}$$

Equation (8) gave the reorder point value for chemicals j

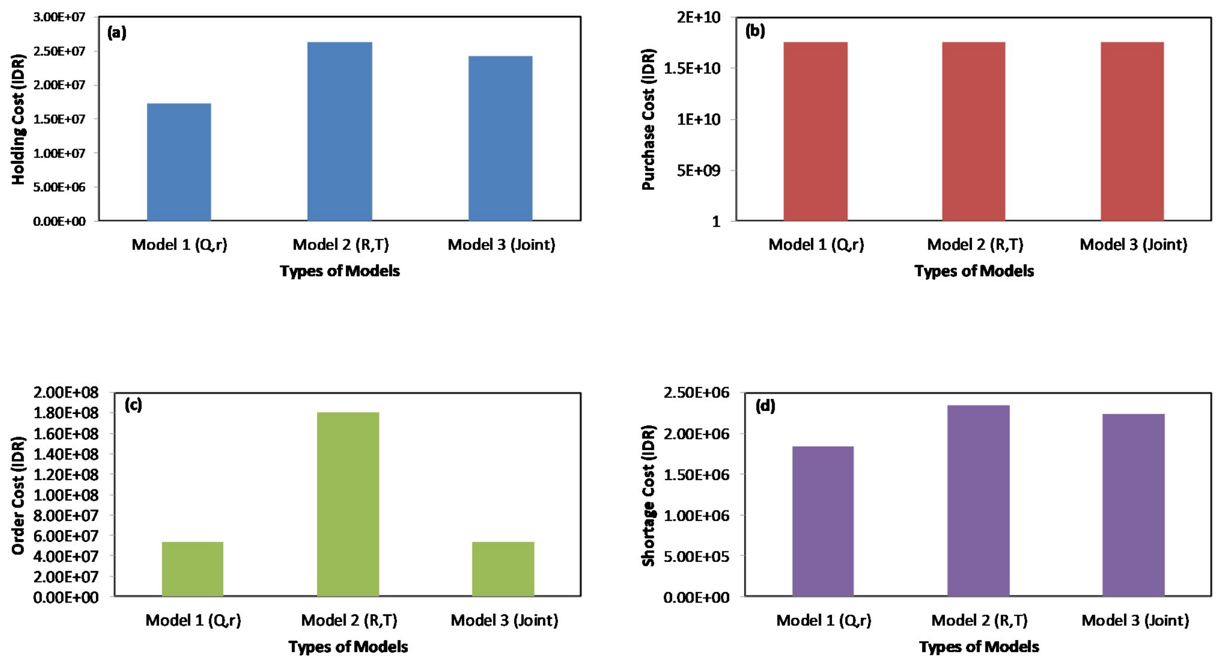


Figure 2. Comparison of Cost (a) Hold, (b) Purchase, (c) Order, and (d) Shortage for Each Inventory Model

(r_j) as follows:

$$r_1 = [-\ln(\alpha_1)\beta_{L_1}] + \gamma_{L_1} \tag{19}$$

$$r_2 = \theta_{L_2}(\text{sc.gammaincinv}[\lambda_{L_2}, (1-\alpha_2)]) \tag{20}$$

$$r_3 = \theta_{L_3}(\text{sc.gammaincinv}[\lambda_{L_3}, (1-\alpha_3)]) \tag{21}$$

$$r_4 = e^{\ln(\eta_{L_4}) - \frac{\ln(\ln(\alpha_4))}{\beta_{L_4}}} \tag{22}$$

Calculating the value of r_j for $j = 2, 3$ in Equations (20) and (21) is based on sc.gammaincinv$[\lambda_{L_j}, (1-\alpha_j)]$, which is the inverse of the under incomplete gamma function for the Shape parameter j (λ_{L_j}) and cumulative probability j $(1-\alpha_j)$, then multiplying the result by the Scale parameter j (θ_j). Sc.gammaincinv is a function from the SciPy module in Python, used to compute the inverse of the incomplete gamma function.

For Model 2, the development of the individual filling time formulation for chemicals j (T_{2j}) from Equation (9) was:

For Model 3, the development of the formulation for the filling time with chemicals (T_{Joint}) in Equation (10) was:

$$T^2_{\text{Joint}} = \Bigg[2A_2^* + 2C_{u_1}\int_{R_1}^{\infty}(z_1 - R_1)\left(\frac{1}{\beta_{(L+T_{o1})_1}} e^{-\frac{(z_1-\gamma_{(L+T_{o1})_1})}{\beta_{(L+T_{o1})_1}}}\right)dz_1 + \sum_{i=2}^{3} 2C_{u_i}\int_{R_i}^{\infty}(z_i - R_i)\, \frac{z_i^{\lambda_{(L+T_{oi})_i}-1} e^{-z_i/\theta_{(L+T_{oi})_i}}}{\Gamma(\lambda_{(L+T_{oi})_i})\theta_{(L+T_{oi})_i}^{\lambda_{(L+T_{oi})_i}}} dz_i + 2C_{u_4}\int_{R_4}^{\infty}(z_4 - R_4)\, \frac{\beta_{(L+T_{o4})_4}}{\eta_{(L+T_{o4})_4}}\left(\frac{z_4}{\eta_{(L+T_{o4})_4}}\right)^{\beta_{(L+T_{o4})_4}-1} e^{-\left(\frac{z_4}{\eta_{(L+T_{o4})_4}}\right)^{\beta_{(L+T_{o4})_4}}} dz_4\Bigg] \times \left(\frac{1}{\sum_{j=1}^{4} h_j D_j}\right)$$

Where T_{oj} is the initial T for chemicals j.

The results obtained are presented in Tables 4 - 6.

In Table 4, the total cost given by Model 1 was $Tc(Q, r) = IDR17,616,779,765.17$, with the reorder-point for chemicals j indicated by their r_{1j} value. The ordering point for liquid aluminum sulfate chemicals was $r_{11} = 75,955.75$ kg, with each order lot being $Q_{11} = 26,653.19$ kg, and similar parameters were observed for other chemicals.

Table 6 provided the results for Model 3, namely the joint replenishment model. The time interval for filling together all chemicals was $T_{Joint} = 0.0017 years$, with total cost generated $Tc(R_j, T_j) = IDR17,624,355,227.64$.

Based on Tables 4 to 6, Model 1 provided a more optimal inventory management policy compared to Models 2 and 3. Holding, ordering, and shortage cost for Model 1 were the lowest compared to Models 2 and 3. The purchase cost for all three models was the same because the number of units

$$T_1 = \sqrt{\frac{2A_2^* + 2C_{u_1}\int_{R_1}^{\infty}(z_1 - R_1)\left(\frac{1}{\beta_{(L+T_{o1})_1}} e^{-\frac{\left(z_1 - \gamma_{(L+T_{o1})_1}\right)}{\beta_{(L+T_{o1})_1}}}\right)dz_1}{h_1 D_1}}$$

$$T_2 = \sqrt{\frac{2A_2^* + 2C_{u_2}\int_{R_2}^{\infty}(z_2 - R_2)\frac{z_2^{\lambda_{(L+T_{o2})_2}-1} e^{-z_2/\theta_{(L+T_{o2})_2}}}{\Gamma(\lambda_{(L+T_{o2})_2})\theta_{(L+T_{o2})_2}^{\lambda_{(L+T_{o2})_2}}}dz_2}{h_2 D_2}}$$

$$T_3 = \sqrt{\frac{2A_2^* + 2C_{u_3}\int_{R_3}^{\infty}(z_3 - R_3)\frac{z_3^{\lambda_{(L+T_{o3})_3}-1} e^{-z_3/\theta_{(L+T_{o3})_3}}}{\Gamma(\lambda_{(L+T_{o3})_3})\theta_{(L+T_{o3})_3}^{\lambda_{(L+T_{o3})_3}}}dz_3}{h_3 D_3}}$$

$$T_4 = \sqrt{\frac{2A_2^* + 2C_{u_4}\int_{R_4}^{\infty}(z_4 - R_4)\left(\frac{\beta_{(L+T_{o4})_4}}{\eta_{(L+T_{o4})_4}}\left(\frac{z_4}{\eta_{(L+T_{o4})_4}}\right)^{\beta_{(L+T_{o4})_4}-1} e^{-\left(\frac{z_4}{\eta_{(L+T_{o4})_4}}\right)^{\beta_{(L+T_{o4})_4}}}\right)dz_4}{h_4 D_4}}$$

to be supplied was equal, based on the average distribution of demand probabilities.

The purchase cost was the product of the average quantity of chemicals purchased in a year and the price of chemicals, which was the same for Models 1, 2, and 3.

Model 1 provided the most optimal inventory policy with the minimum total cost, as shown in Figure 1. The difference in total cost between model 1 and models 2 and 3 is IDR 136,711,926.16 and IDR 7,575,462.47, respectively.

Figure 2 showed that for the holding cost, purchase and shortage cost components, the model 1 provided the smallest value.

4. CONCLUSIONS

In conclusion, the results obtained were as follows: The time series data for the demand for chemicals at PDAM Tirta Musi from 2016 to 2023 had different probability distribution. Liquid aluminum sulfate followed an Exponential distribution, chlorine gas, and lime powder followed Gamma probability distribution, while calcium hypochlorite had Weibull probability distribution. The appropriate inventory model for the management of chemicals inventory was Model 1, namely probability (Q, r) model with individual replenishment because it provided a smaller total cost compared to both (R, T) model with individual replenishment and the joint replenishment model. For further improvement of the research, it is suggested to combine individual and joint replenishment in one inventory model and building a dynamic probabilistic inventory model for multi-items.

5. ACKNOWLEDGMENT

This study was part of the Doctoral Dissertation of Mathematics and Natural Sciences Doctoral Studies in the speciality of Mathematics at Sriwijaya University. The authors are grateful to Sriwijaya University, all supervisors, the Main Director and the staff at PDAM Tirta Musi Palembang who provided assistance during this study. The research publication of this article was funded by DIPA of Kementerian Pendidikan, Kebudayaan, Riset, dan Teknologi tahun 2024. Nomor SP DIPA-023.17.1.690523/2024 revision-1, On February 04, 2024. In accordance with the Rector's Decree Number 090/E5/PG.02.0 0.Pl/2024, on June 11, 2024.

REFERENCES

Ayu, D., S. Fauji, and H. Purnomo (2022). Controlling Poly Aluminum Cloraide Inventory with the Economic Order Quantity Method at PDAM Nganjuk Regency. In *Management and Business Symposium*, volume 1. pages 152–159

Bahagia, S. N. (2006). *Inventory System*. ITB Press, first edition

Castellano, D. and L. C. Santillo (2023). Stochastic Joint Replenishment Problem under a Fill Rate Constraint with Controllable Lead Times and Shared Cost Allocation. *Operations Research Letters*, **51**(4); 385–392

Christou, I. T., K. Skouri, and A. G. Lagodimos (2020). Fast Evaluation of a Periodic Review Inventory Policy. *Computers and Industrial Engineering*, **144**; 106389

Das, K. and S. Islam (2024). A Deterministic Multi-Item Inventory Model with Quadratic Demand under Neutrosophic and Pythagorean Hesitant Fuzzy Programming Approach. *Results in Control and Optimization*, **14**; 100367

De, S. K. (2021). Solving an EOQ Model under Fuzzy Reasoning. *Applied Soft Computing*, **99**; 106892

de Figueiredo, M. P. S., G. R. Moreira, C. C. R. de Brito, F. Gomes-Silva, A. L. Pinto dos Santos, M. L. L. da Costa, M. C. Filho, and L. Silva do Amaral (2023). Method to Generate Growth and Degrowth Models Obtained from Existing Models Compositions Applied to Animal Sciences – The Athens-Canadian Chicken Growth Case. *Livestock Science*, **269**; 105162

Dewi, D. C., E. R. EW, and S. Suganda (2019). Planning and Controlling Liquid Aluminum Sulfate Inventory with the EOQ Method Case Study at PDAM Tirta Mayang Jambi City. *Journal of Science and Technology*, **2**; 121

Dolgan, N. C., L. M. Haidar, B. Esmer, and M. Y. Jaber (2020). Temporary Price Increase during Replenishment Lead Time. *Applied Mathematical Modelling*, **78**; 217–231

Fitria, D., M. Pamuji, and M. Pompa (2015). Pump Motor Inverter at PDAM Tirta Musi Palembang. *Journal of Technology Dissemination*, **3**(1); 46–55

Gutierrez, M. and F. A. Rivera (2021). Undershoot and Order Quantity Probability Distributions in Periodic Review, Reorder Point, Order-Up-to-Level Inventory Systems with Continuous Demand. *Applied Mathematical Modelling*, **91**; 791–814

Hansen, O., S. Transchel, and H. Friedrich (2022). Replenishment Strategies for Lost Sales Inventory Systems of Perishables under Demand and Lead Time Uncertainty. *European Journal of Operational Research*, **308**(2); 661–675

Lesmono, D. and T. Limansyah (2017). A Multi Item Probabilistic Inventory Model. *Journal of Physics: Conference Series*, **893**(1); 012024

Lesmono, D. and T. Limansyah (2019). A Probabilistic Inventory Model with Deterioration Factor and Continuous Discount Function. *AIP Conference Proceedings*, **2192**(1); 060013

Ophokenshi, N. P., C. W. I. Emmanuel, and M. O. Sadik (2019). Analysis of an Inventory System for Items with Stochastic Demand and Time Dependent Three-Parameter Weibull Deterioration Function. *Applied Mathematics*, **10**(9); 728–742

Otsu, T. and G. Taniguchi (2020). Kolmogorov–Smirnov Type Test for Generated Variables. *Economics Letters*, **195**; 109401

Rojano, A. P., A. Andrea, M. P. Polanco, M. S. Jiménez, and L. D. Rosa (2020). An Optimization Approach for Inventory Costs in Probabilistic Inventory Models: A Case Study. *Revista Chilena de Ingenieria*, **28**; 383–395

Saha, C., D. K. Jana, and A. Duary (2023). Enhancing Production Inventory Management for Imperfect Items Using Fuzzy Optimization Strategies and Differential Evolution (DE) Algorithms. *Franklin Open*, **5**; 100051

Salas-Navarro, K., J. Acevedo-Chedid, G. M. Árquez, W. F. Florez, H. Ospina-Mateus, S. S. Sana, and L. E. Cárdenas-Barrón (2020). An EPQ Inventory Model Considering an Imperfect Production System with Probabilistic Demand and Collaborative Approach. *Journal of Advances in Management Research*, **17**(2); 282–304

Sayed, A. I. A. and S. S. R. Muhammad (2022). A Simulation Study on The Simulated Annealing Algorithm in Estimating The Parameters of Generalized Gamma Distribution. *Science and Technology Indonesia*, **7**(1); 84–90

Silitonga, R. Y. H., L. R. Kristiana, and T. A. Parley (2021). A Multi-Item Probabilistic Inventory Model that Considers Expiration Factor, All Unit Discount Policy and Warehouse Capacity Constraints. *Journal of Industrial Engineering*, **23**(2); 139–148

Silver, E., D. Pyke, and D. Thomas (2017). *Inventory Management and Production Planning Scheduling*. CRC Press, fourth edition

Susanti, E., F. M. Puspita, S. S. Supadi, E. Yuliza, and A. F. Ramadhan (2023). Improve Fuzzy Inventory Model of Fractal Interpolation with Vertical Scaling Factor. *Science and Technology Indonesia*, **8**; 654–659

Sutoni, A. and D. H. Taufik (2019). Inventory Planning with Method Q and Method P for Probabilistic Demand on Chrysanthemum Seeds at PT Transplants Indonesia. *International Journal of Innovation in Enterprise System*, **3**(01); 38–46

Wan, G., Y. Cao, and S. Wang (2022). Evaluation of a Joint Replenishment Policy with Sales-Based Threshold and Stochastic Demand. *Computers and Industrial Engineering*, **172**; 108525

Wang, L. and H. Chen (2022). Optimization of a Stochastic Joint Replenishment Inventory System with Service Level Constraints. *Computers and Operations Research*, **148**; 106001

Zhang, J., L. Deng, H. Liu, and T. C. E. Cheng (2022). Which Strategy Is Better for Managing Multi-Product Demand Uncertainty: Inventory Substitution or Probabilistic Selling? *European Journal of Operational Research*, **302**(1); 79–95

