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Research Paper

The Partition Dimension of Daisy Graphs and Its Barbell

Asmiati[1*], Aulia Ajie Rachmawati[1], Dina Eka Nurvazly[1], Nur Hamzah[1]

[1]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Universitas Lampung, Bandar Lampung, 35145, Indonesia*
*Corresponding author: asmiati.1976@fmipa.unila.ac.id

Abstract
The partition dimension of a graph is determined by minimum number of vertex partitions such that every vertex has different distances to the ordered partitions. A complete graph is very easy to determine its partition dimension because each vertex has the same distance to other vertices. However, what are the partition dimension if a complete graph is modified so that it becomes a daisy graph. In this paper, we discuss the partition dimension of daisy graphs. Next, we will also provide barbell graph operations on daisy graphs.

Keywords
Partition Dimension, Daisy Graph, Barbell Graph

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1. INTRODUCTION

The concept of partition dimension was firstly introduced by Chartrand as an extension of the metric dimension concept. Since its introduction, numerous studies have focused on determining the partition dimension of various graphs.

Let $G = (V, E)$ be a connected graph and $\Pi = \{S_1, S_2, \ldots, S_k\}$ partition of $V(G)$. The representation of v with respect to Π is the k-vector $r(v|\Pi) = (d(v, S_1), d(v, S_2), \ldots, d(v, S_k))$, where $d(v, S_i) = \min_{w \in S_i} d(v, w)$ for $1 \leq i \leq k$. If $r(v|\Pi)$ are distinct for every $v \in V(G)$, then Π is a resolving with k partition. Next, the smallest k is known as the partition dimension of G, denoted by $pd(G)$ (Chartrand et al., 2000).

Let K_n be a complete graph with vertices $\{v_i|i \in [1.n]\}$. The Daisy graph, denoted by $D(K_n)$, $n \geq 3$ is a graph constructed from K_n and n vertices $\{w_i|i \in [1.n]\}$ such that vertices v_i and $v_{(i+1)}$ are adjacent to w_i for $i \in [1.n]$ where $v_{(n+1)} = v_1$ (Sugeng et al., 2022). The barbell daisy graph, denoted by $B_{D(K_n)}$, is obtained by copying a daisy graph (that is, $D'(K_n)$) and connecting two graphs with a bridge (Asmiati et al., 2018). We assume a vertex set of $D'(K_n)$ is $\{v'_i, w'_i|1 \leq i \leq n\}$ and a bridge in $B_{D(K_n)}$ connecting $(v_1 v'_1)$.

Many researchers have obtained partition dimensions for several classes of graphs. Asmiati (2012) obtained partition dimension of amalgamation of stars, Fernau et al. (2014) for unicyclic graphs, and Amrullah et al. (2015) succeeded for a subdivision of complete graph. Grigorious et al. (2014) obtains the partition dimensions of a class of circular graphs and is continued by Maritz and Vetrik (2018). Rodríguez-Velazquez et al. (2014) found partition dimension of trees and then developed by Bagus and Baskoro (2015). Next, Fredlina and Baskoro (2015) for some families of trees.

The research topic of partition dimension of graphs is still interesting today, not only in certain graph classes but also in graph operations. Haryeni et al. (2017) determined partition dimension of disconnected graphs and Amrullah et al. (2019) for subdivision graph on the star. Next, Amrullah (2020) got a subdivision of a homogeneous firecracker. Mohan et al. (2019) examined the partition dimension of series-parallel graphs, highlighting its computational complexity and structural properties, while Haryeni et al. (2019) found a method to construct graphs with certain partition dimension and Hernando et al. (2019) determined resolving dominating partitions in graphs.

Other interesting results for dimension partition of graphs that have been obtained are Baskoro and Haryeni (2020) determined all graphs of order $n \geq 11$ and diameter 2 with partition dimension $(n - 3)$. Khali et al. (2021) focused on bounded partition dimensions in convex polytopes with pendant edges, providing bounds and constructions for these families. Similarly, Azeem et al. (2022) investigated for hexagonal Möbius ladders, deriving sharp bounds that demonstrate the parameter's sensitivity to symmetrical structures. Kuziak et al. (2023) determined new definition about the edge partition dimension of graphs, whereas Ridwan et al. (2023) determined the dominating partition dimension and locating chromatic number of graphs.

The latest research on partition dimension was conducted by Hafidh and Baskoro (2024) used palm approach to determine partition dimension of trees. Koam et al. (2024) analyzed cardinality bounds for partition resolving sets in convex polytope-like graphs. Similarly, Bhatti et al. (2024) studied the partition dimension in generalized hexagonal cellular networks, emphasizing its utility in wireless communication systems.

Based on literature studies, there is no discussion about the partition dimensions of daisy graphs. Likewise for the partition dimensions of the results of barbell graph operations. This research focuses on determining the partition dimension of both the daisy graph and the barbell daisy graph.

The following theorem will be used for the next discussion which is taken from Chartrand et al. (2000)

Theorem 1.1 *The partition dimension of cycle graph C_n for $n \geq 3$ is 3.*

2. METHODS

In this section, steps are given to determine partition dimension of daisy graphs and its barbell.

1. Construct daisy graphs $D(K_n)$ for $n \geq 3$.
2. Determine the lower bound of partition dimension for daisy graph $D(K_n)$ for $n \geq 3$. Since daisy graph contain cycles, then by Theorem 1.1. we have, $pd(D(K_n)) \geq 3$.
3. Determine the upper bound of partition dimension for daisy graph $D(K_n)$ for $n \geq 3$. Every vertices of $V(D(K_n))$ for $n \geq 3$ are grouped into different partition classes so that they become a minimum resolving partition set.
4. The partition dimension of the daisy graph is determined from the lower and upper bound values obtained.
5. These steps are also carried out to determine the partition dimension of the barbell daisy graph $B_{D(K_n)}$ for $n \geq 3$. However, since the barbell daisy graph contains a daisy graph, then $pd(B_{D(K_n)}) \geq pd(D(K_n))$.

3. RESULTS AND DISCUSSION

In this section, we determine the partition dimension of Daisy graph and barbell Daisy graph.

Lemma 3.1 *Suppose $\Pi = \{S_1, S_2, \ldots, S_k\}$ resolving partition of Daisy graph $D(K_n)$, where $n \geq 3$ and each S_k contains at least one vertex $v_i \in V(K_n)$. If $v_i, v_j \in D(K_n)$, with $i \neq j$ belong to the same partition, then $r(v_i|\Pi) = r(v_j|\Pi)$.*

Proof. Since S_k contains at least one vertex $v_i \in V(K_n)$, then $d(v_i, S_k) = 1$. If $v_i, v_j \in D(K_n)$, where $i \neq j$ and they belong to the same partition, in their representation, the value 0 lies on the same ordinate. Thus, $r(v_i|\Pi) = r(v_j|\Pi)$.

Lemma 3.2 *Let the daisy path $l_{(w_i,w_j)} : w_i, \ldots, w_j$, where $i \neq j$, passing through three consecutive vertices in K_n. Let $\Pi = \{S_1, S_2, \ldots, S_k\}$ be a resolving partition of Daisy graph $D(K_n)$, where $n \geq 3$ and there exists a path $l_{(w_i,w_j)}$. If v_i belongs to the t partition $S_1, S_2, \ldots, S_t$, where $t = \lceil \frac{n}{3} \rceil$, $t < k$, then three consecutive vertices on $l_{(w_i,w_j)}$ are in the same partition, and w_i and w_j are in different singleton partitions.*

Proof. Let $V(l_{(w_i,w_j)}) = \{w_i, v_i, v_{i+1}, v_{i+2}, w_j\}$. Furthermore, since Π is a resolving partition, then $r(v_i, w_i|\Pi)$ are distinct. It will be proven that if three consecutive vertices on the path $l_{(w_i,w_j)}$, namely v_i, v_{i+1}, v_{i+2}, are not in the same partition or w_i, w_j are not in different singleton partitions, then v_i is in more than t partition with $t = \lceil n/3 \rceil$.

Three consecutive vertices on the path $l_{(w_i,w_j)}$ lie in different partition. These three vertices can be divided into either two or three partitions. Therefore, there are at least $\lceil \frac{n}{3} \rceil - 1 + 2 = \lceil \frac{n}{3} \rceil + 1 \geq t = \lceil \frac{n}{3} \rceil$. Next, w_i and w_j on the path $l_{(w_i,w_j)}$ do not belong to different singleton partition. Since it is a path, vertex w_i is adjacent to v_i and vertex v_{i+2} is adjacent to w_j. $d(v_i, w_i) = d(v_{i+2}, w_j) = 1$, $d(v_i, w_j) = d(v_{i+2}, w_i) = 2$, and every vertex v_k, $k = 1, 2, \ldots, n$, is adjacent to each other. To ensure different representations of vertices, at least v_i and v_{i+2} in this condition must belong to different partitions. Thus, there are at least $\lceil \frac{n}{3} \rceil + 1 \geq t = \lceil \frac{n}{3} \rceil$ partition.

Theorem 3.1 *The partition dimension of Daisy graph $pd(D(K_n))$ is :*

$$pd(D(K_n)) = \begin{cases} 3 & \text{for } n = 3 \\ n - \lfloor \frac{n}{3} \rfloor & \text{for } n > 3 \end{cases}$$

Proof. The proof is divided into four cases

Case 1. For $n = 3$.
The graph $D(K_3)$ contains cycle graphs, based on Theorem 1.1, the lower bound for the partition dimension of the Daisy graph $D(K_3)$:

$$pd(D(K_3)) \geq 3 \tag{1}$$

Given $\Pi = \{S_1, S_2, S_3\}$, be the partition of the $V(D(K_3))$ with $S_1 = \{v_1, v_2, w_1\}$, $S_2 = \{v_3, w_2\}$, $S_3 = \{w_3\}$. The representative for $V(D(K_3))$ with respect to Π are :

$$\begin{aligned} r(v_1|\Pi) &= (0, 1, 1) \\ r(v_2|\Pi) &= (0, 1, 2) \\ r(v_3|\Pi) &= (1, 0, 1) \\ r(w_1|\Pi) &= (0, 2, 2) \\ r(w_2|\Pi) &= (1, 0, 2) \\ r(w_3|\Pi) &= (1, 1, 0) \end{aligned}$$

Since every representative are distinct, Π is resolving partition of $D(K_3)$. Then the lower bound for the partition dimension of the Daisy graph $D(K_3)$:

$$pd(D(K_3)) \leq 3 \tag{2}$$

Based on (1) and (2), we have $pd(D(K_3)) = 3$.
Case 2. For $n > 3$, with $n \equiv 1(mod 3)$.
To establish the lower bound of the Daisy graph $D(K_n)$ with $n > 3$ and $n \equiv 1 \mod 3$, it will be shown that $(n - \lfloor \frac{n}{3} \rfloor - 1)$

partitions are insufficient. Assume there are $(n - \lfloor\frac{n}{3}\rfloor - 1)$ partitions in $D(K_n)$. According to Lemma 3.2, a total of $3k$ vertices, with $k = (\lfloor\frac{n}{3}\rfloor - 1)$ being a constant, from n vertices v_i are contained in k partition. Thus, three consecutive vertices v_i on the path $l_{(w_i,w_j)}$ are in the same partition class, and there are at least $(k+1)$ distinct vertices w_j on the path $l_{(w_i,w_j)}$ with w_j as the sole member of a different partition. Then, the remaining four vertices v_i are partitioned into two partition classes. Hence, there are at least $k + (k+1) + 2 = (n - \lfloor\frac{n}{3}\rfloor)$ partition classes, contradicting the assumption. Therefore,

$$pd(D(K_n)) \geq (n - \lfloor\frac{n}{3}\rfloor) \tag{3}$$

Given $\Pi = \{S_1, S_2, \ldots S_{(n-\lfloor\frac{n}{3}\rfloor)}\}$, be the partition of the $V(D(K_n))$ with :

$$\begin{aligned} S_{(n-\lfloor\frac{n}{3}\rfloor-1)} &= \{v_i | i = n, (n-1)\} \\ S_{(n-\lfloor\frac{n}{3}\rfloor-2)} &= \{v_i | i = (n-2), (n-3)\} \\ S_1 &= \{v_i | i = 1, 2, 3\} \text{ if } n \geq 7 \\ S_2 &= \{v_i | i = 4, 5, 6\} \text{ if } n \geq 10 \\ S_3 &= \{v_i | i = 7, 8, 9\} \text{ if } n \geq 13 \\ S_{\frac{2}{3}i} &= \{w_i | i \equiv 0 \pmod 3\} \\ S_{\lceil\frac{2}{3}i\rceil} &= \{w_i | i \equiv 1 \pmod 3\} \\ S_{\lfloor\frac{2}{3}i\rfloor} &= \{w_i | i \equiv 2 \pmod 3\} \end{aligned}$$

Thus, w_i when $i = n$ is an element of partition-$\lceil\frac{2}{3}i\rceil = (n - \lfloor\frac{n}{3}\rfloor)$. Consequently, the vertices of Daisy graph $D(K_n)$ have distinct representations, hence Π is a resolving partition. The upper bound of $pd(D(K_n))$ is

$$pd(D(K_n)) \leq (n - \lfloor\frac{n}{3}\rfloor) \tag{4}$$

Based on (3) and (4), we have $pd(D(K_n)) = (n - \lfloor\frac{n}{3}\rfloor)$
Case 3. For $n > 3$, with $n \equiv 2 (mod 3)$.
To establish the lower bound of the Daisy graph $D(K_n)$ with $n > 3$ and $n \equiv 2 \mod 3$, it will be shown that $(n - \lfloor\frac{n}{3}\rfloor - 1)$ partitions are insufficient. Assume there are $(n - \lfloor\frac{n}{3}\rfloor - 1)$ partitions in $D(K_n)$. According to Lemma 3.2, n vertices of v_i are contained in $\lceil\frac{n}{3}\rceil$ partition. Thus, three consecutive vertices v_i on the path $l_{(w_i,w_j)}$ are in the same partition, and there are at least $\lceil\frac{n}{3}\rceil$ distinct vertices w_j on the path $l_{(w_i,w_j)}$ with w_j as the sole member of a different partition. Hence, there are at least $\lceil\frac{n}{3}\rceil + \lceil\frac{n}{3}\rceil = (n - \lfloor\frac{n}{3}\rfloor)$ partition, contradicting the assumption. Therefore,

$$pd(D(K_n)) \geq (n - \lfloor\frac{n}{3}\rfloor) \tag{5}$$

Given $\Pi = \{S_1, S_2, \ldots S_{(n-\lfloor\frac{n}{3}\rfloor)}\}$, be the partition of the $V(D(K_n))$ with :

$$\begin{aligned} S_{(n-\lfloor\frac{n}{3}\rfloor-1)} &= \{v_i | i = n, (n-1)\} \\ S_1 &= \{v_i | i = 1, 2, 3\} \text{ if } n \geq 5 \\ S_3 &= \{v_i | i = 4, 5, 6\} \text{ if } n \geq 8 \\ S_5 &= \{v_i | i = 7, 8, 9\} \text{ if } n \geq 11 \\ S_{n-\lfloor\frac{n}{3}\rfloor} &= \{w_i | i = n\} \\ S_{\frac{2}{3}i} &= \{w_i | i \equiv 0 \pmod 3\} \\ S_{\lceil\frac{2}{3}i\rceil} &= \{w_i | i \equiv 1 \pmod 3\} \\ S_{\lfloor\frac{2}{3}i\rfloor} &= \{w_i | i \equiv 2 \pmod 3\} \end{aligned}$$

Thus, w_i when $i = n$ is an element of partition-$(n - \lfloor\frac{n}{3}\rfloor)$. Consequently, the vertices of Daisy graph $D(K_n)$ have distinct representations, hence Π is a resolving partition. The upper bound of $pd(D(K_n))$ is

$$pd(D(K_n)) \leq (n - \lfloor\frac{n}{3}\rfloor) \tag{6}$$

Based on (5) and (6), we have $pd(D(K_n)) = (n - \lfloor\frac{n}{3}\rfloor)$.
Case 4. For $n > 3$, with $n \equiv 0 (mod 3)$.
To establish the lower bound of the Daisy graph $D(K_n)$ with $n > 3$ and $n \equiv 0 \mod 3$, it will be shown that $(n - \lfloor\frac{n}{3}\rfloor - 1)$ partitions are insufficient. Assume there are $(n - \lfloor\frac{n}{3}\rfloor - 1)$ partitions in $D(K_n)$. According to Lemma 3.2, n vertices of v_i are contained in $\lceil\frac{n}{3}\rceil$ partition. Thus, three consecutive vertices v_i on the path $l_{(w_i,w_j)}$ are in the same partition, and there are at least $\lceil\frac{n}{3}\rceil$ distinct vertices w_j on the path $l_{(w_i,w_j)}$ with w_j as the sole member of a different partition. Hence, there are at least $\lceil\frac{n}{3}\rceil + \lceil\frac{n}{3}\rceil = (n - \lfloor\frac{n}{3}\rfloor)$ partition, contradicting the assumption. Therefore,

$$pd(D(K_n)) \geq (n - \lfloor\frac{n}{3}\rfloor) \tag{7}$$

Given $\Pi = \{S_1, S_2, \ldots S_{(n-\lfloor\frac{n}{3}\rfloor)}\}$, be the partition of the $V(D(K_n))$ with :

$$\begin{aligned} S_{2\lfloor\frac{i}{3}\rfloor-1} &= \{v_i | i = 1, 2, 3, \ldots, n\} \\ S_{\frac{2}{3}i} &= \{w_i | i \equiv 0 \pmod 3\} \\ S_{\lceil\frac{2}{3}i\rceil} &= \{w_i | i \equiv 1 \pmod 3\} \\ S_{\lfloor\frac{2}{3}i\rfloor} &= \{w_i | i \equiv 2 \pmod 3\} \end{aligned}$$

Thus, w_i when $i = n$ is an element of partition-$(\frac{2}{3}i) = (n - \lfloor\frac{n}{3}\rfloor)$. Consequently, the vertices of Daisy graph $D(K_n)$ have distinct representations, hence Π is a resolving partition. The upper bound of $pd(D(K_n))$ is

$$pd(D(K_n)) \leq (n - \lfloor\frac{n}{3}\rfloor) \tag{8}$$

Based on (7) and (8), we have $pd(D(K_n)) = (n - \lfloor\frac{n}{3}\rfloor)$. The complete proof.
Figures 1 and 2 show that the minimum resolving partition for $D(K_3)$ and $D(K_6)$ is 3 and 4, respectively.

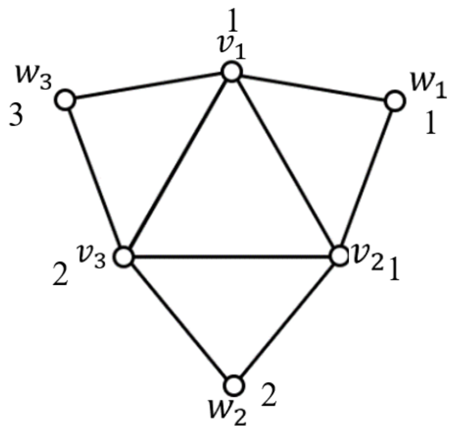

Figure 1. Graph $D(K_3)$ with Minimum Resolving Partition.

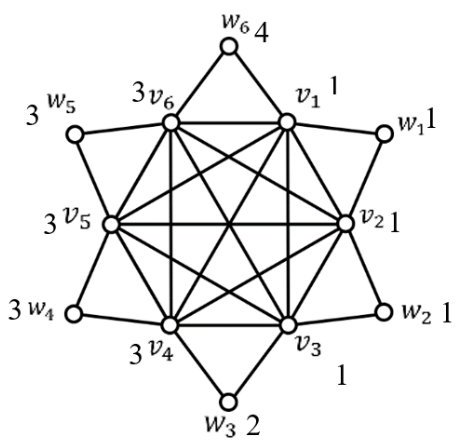

Figure 2. Graph $D(K_6)$ with Minimum Resolving Partition.

Theorem 3.2 *The partition dimension of barbell Daisy graph* $pd(B_{(D(K_n))})$ *is :*

$$pd(B_{(D(K_n))}) = \begin{cases} 4 & \text{for } n = 3, 4 \\ n - \lfloor \frac{n}{3} \rfloor & \text{for } n > 4 \end{cases}$$

Proof. The proof is divided into five cases

Case 1. For $n = 3$.
Since the barbell Daisy graph $B_D(K_3)$, contains Daisy graph $D(K_3)$, according to Theorem 3.1, $pd(B_D(K_3)) \geq 3$. Suppose there are 3 partitions with $r(v_1) = r(v_1')$, then $r(v_1|\Pi) = r(v_1'|\Pi) = (0, 1, 1)$, a contradiction. Thus, at least $3 + 1$ partitions are needed to satisfy the partition dimension requirement of the barbell Daisy graph $B_D(K_3)$. Consequently,

$$pd(B_D(K_3)) \geq 4 \tag{9}$$

Given $\Pi = \{S_1, S_2, S_3, S_4\}$, be the partition of the $V(B_D(K_3))$ with :

$$S_1 = \{v_1, w_1, v_2, v_1'\}$$
$$S_2 = \{w_2, v_3, w_1', v_2', w_2'\}$$
$$S_3 = \{w_3, v_3'\}$$
$$S_4 = \{w_3'\}$$

Thus, each vertex in the barbell Daisy graph $B_D(K_3)$, has a distinct representation, implying that Π is a resolving partition. Consequently,

$$pd(B_D(K_3)) \leq 4 \tag{10}$$

Based on (9) and (10), we have $pd(B_D(K_3)) = 4$.

Case 2. For $n = 4$.
Since the barbell Daisy graph $B_D(K_4)$, contains Daisy graph $D(K_4)$, according to Theorem 3.1, $pd(B_D(K_4)) \geq 3$. Suppose there are 3 partitions with $r(v_1) = r(v_1')$, then $r(v_1|\Pi) = r(v_1'|\Pi) = (0, 1, 1)$, a contradiction. Thus, at least $3 + 1$ partitions are needed to satisfy the partition dimension requirement of the barbell Daisy graph $B_D(K_4)$. Consequently,

$$pd(B_D(K_4)) \geq 4 \tag{11}$$

Given $\Pi = \{S_1, S_2, S_3, S_4\}$, be the partition of the $V(B_D(K_4))$ with :

$$S_1 = \{v_1, w_1, v_2, w_2, v_1', w_1'\}$$
$$S_2 = \{v_3, w_3, v_4, v_2', w_2'\}$$
$$S_3 = \{w_4, v_3', w_3'\}$$
$$S_4 = \{v_4', w_4'\}$$

Thus, each vertex in the barbell Daisy graph $B_D(K_4)$, has a distinct representation, implying that Π is a resolving partition. Consequently,

$$pd(B_D(K_4)) \leq 4 \tag{12}$$

Based on (11) and (12), we have $pd(B_D(K_4)) = 4$.

Case 3. For $n > 4$, with $n \equiv 1(mod 3)$.
Since the barbell Daisy graph $B_D(K_n)$, contains the Daisy graph $D(K_n)$, according to Theorem 3.1,

$$pd(B_{D(K_n)}) \geq (n - \lfloor \frac{n}{3} \rfloor) \tag{13}$$

Given $\Pi = \{S_1, S_2, \ldots, S_{(n-\lfloor \frac{n}{3} \rfloor)}\}$, be the partition of the

$V(B_{D(K_n)})$ with :

$$\begin{aligned} S_{(n-\lfloor\frac{n}{3}\rfloor-1)} &= \{v_i \mid i = n, (n-1)\} \\ S_{(n-\lfloor\frac{n}{3}\rfloor-2)} &= \{v_i \mid i = (n-2), (n-3)\} \\ S_1 &= \{v_i \mid i = 1, 2, 3 \text{ if } n \geq 7\} \\ S_2 &= \{v_i \mid i = 4, 5, 6 \text{ if } n \geq 10\} \\ S_3 &= \{v_i \mid i = 7, 8, 9 \text{ if } n \geq 13\} \end{aligned}$$

$$\begin{aligned} S_{\frac{2}{3}i} &= \{w_i \mid i \equiv 0 \pmod 3\} \\ S_{\lceil\frac{2}{3}i\rceil} &= \{w_i \mid i \equiv 1 \pmod 3\} \\ S_{\lfloor\frac{2}{3}i\rfloor} &= \{w_i \mid i \equiv 2 \pmod 3\} \end{aligned}$$

$$\begin{aligned} S_2 &= \{v'_i \mid i = n, (n-1)\} \\ S_3 &= \{v'_i \mid i = (n-2), (n-3)\} \\ S_{(n-\lfloor\frac{n}{3}\rfloor)} &= \{v'_i \mid i = 1, 2, 3 \text{ if } n \geq 7\} \\ S_{(n-\lfloor\frac{n}{3}\rfloor-2)} &= \{v'_i \mid i = 4, 5, 6 \text{ if } n \geq 10\} \\ S_{((n-\lfloor\frac{n}{3}\rfloor-4))} &= \{v'_i \mid i = 7, 8, 9 \text{ if } n \geq 13\} \end{aligned}$$

$$\begin{aligned} S_{\lceil\frac{2}{3}(n-i)\rceil+1} &= \{w'_i \mid i \equiv 0 \pmod 3, i \equiv 2 \pmod 3\} \\ S_{(\frac{2}{3}(n-i))+1} &= \{w'_i \mid i \equiv 1 \pmod 3\} \end{aligned}$$

Thus, each vertex of barbell Daisy graph $B_{D(K_n)}$, has a distinct representation, implying that Π is a resolving partition. Therefore, the upper bound is

$$pd(B_{D(K_n)}) \leq (n - \lfloor\frac{n}{3}\rfloor) \tag{14}$$

Based on (13) and (14), we have $pd(B_{D(K_n)}) = (n - \lfloor\frac{n}{3}\rfloor)$.

Case 4. For $n > 4$, with $n \equiv 2(mod 3)$.
Since the barbell Daisy graph $B_D(K_n)$, contains the Daisy graph $D(K_n)$, according to Theorem 3.1,

$$pd(B_{D(K_n)}) \geq (n - \lfloor\frac{n}{3}\rfloor) \tag{15}$$

Given $\Pi = \{S_1, S_2, \ldots, S_{(n-\lfloor\frac{n}{3}\rfloor)}\}$, be the partition of the $V(B_{D(K_n)})$ with :

$$\begin{aligned} S_{(n-\lfloor n/3\rfloor-1)} &= \{v_i \mid i = n, (n-1)\} \\ S_1 &= \{v_i \mid i = 1, 2, 3 \text{ if } n \geq 5\} \\ S_3 &= \{v_i \mid i = 4, 5, 6 \text{ if } n \geq 8\} \\ S_5 &= \{v_i \mid i = 7, 8, 9 \text{ if } n \geq 11\} \end{aligned}$$

$$\begin{aligned} S_{(n-\lfloor n/3\rfloor)} &= \{w_i \mid i = n\} \\ S_{(\frac{2}{3}i)} &= \{w_i \mid i \equiv 0 \pmod 3\} \\ S_{(\lceil\frac{2}{3}i\rceil)} &= \{w_i \mid i \equiv 1 \pmod 3\} \\ S_{(\lfloor\frac{2}{3}i\rfloor)} &= \{w_i \mid i \equiv 2 \pmod 3\} \end{aligned}$$

$$\begin{aligned} S_2 &= \{v'_i \mid i = n, (n-1)\} \\ S_{(n-\lfloor n/3\rfloor)} &= \{v'_i \mid i = 1, 2, 3 \text{ if } n \geq 5\} \\ S_{(n-\lfloor n/3\rfloor-2)} &= \{v'_i \mid i = 4, 5, 6 \text{ if } n \geq 8\} \\ S_{(n-\lfloor n/3\rfloor-4)} &= \{v'_i \mid i = 7, 8, 9 \text{ if } n \geq 11\} \end{aligned}$$

$$\begin{aligned} S_1 &= \{w'_i \mid i = n\} \\ S_{(\lfloor\frac{2}{3}(n-i)\rfloor+2)} &= \{w'_i \mid i \equiv 0 \pmod 3, i \equiv 1 \pmod 3\} \\ S_{(\frac{2}{3}(n-i)+2)} &= \{w'_i \mid i \equiv 2 \pmod 3\} \end{aligned}$$

Thus, each vertex of barbell Daisy graph $B_{D(K_n)}$, has a distinct representation, implying that Π is a resolving partition. Therefore, an upper bound on $pd(B_{D(K_n)})$ is

$$pd(B_{D(K_n)}) \leq (n - \lfloor\frac{n}{3}\rfloor) \tag{16}$$

Based on (15) and (16), we have $pd(B_{D(K_n)}) = (n - \lfloor\frac{n}{3}\rfloor)$.
Case 5. For $n > 4$, with $n \equiv 0(mod 3)$.
Since the barbell Daisy graph $B_D(K_n)$, contains the Daisy graph $D(K_n)$, according to Theorem 3.1,

$$pd(B_{D(K_n)}) \geq (n - \lfloor\frac{n}{3}\rfloor) \tag{17}$$

Given $\Pi = \{S_1, S_2, \ldots, S_{(n-\lfloor\frac{n}{3}\rfloor)}\}$, be the partition of the $V(B_{D(K_n)})$ with :

$$\begin{aligned} S_{(2\lceil i/3\rceil-1)} &= \{v_i \mid i = 1, 2, 3, \ldots, n\} \\ S_{(\frac{2}{3}i)} &= \{w_i \mid i \equiv 0 \pmod 3\} \\ S_{(\lceil\frac{2}{3}i\rceil)} &= \{w_i \mid i \equiv 1 \pmod 3\} \\ S_{(\lfloor\frac{2}{3}i\rfloor)} &= \{w_i \mid i \equiv 2 \pmod 3\} \end{aligned}$$

$$\begin{aligned} S_{(2\lceil(n-i+1)/3\rceil)} &= \{v'_i \mid i = 1, 2, 3, \ldots, n\} \\ S_{(\lceil\frac{2}{3}(n-i)\rceil+1)} &= \{w'_i \mid i \equiv 2 \pmod 3\} \\ S_{(\lfloor\frac{2}{3}(n-i)\rfloor+1)} &= \{w'_i \mid i \equiv 1 \pmod 3\} \\ S_{(\frac{2}{3}(n-i)+2)} &= \{w'_i \mid i \equiv 0 \pmod 3\} \end{aligned}$$

Thus, each vertex of barbell Daisy graph $B_{D(K_n)}$, has a distinct representation, implying that Π is a resolving partition. Therefore, an upper bound on $pd(B_{D(K_n)})$ is

$$pd(B_{D(K_n)}) \leq (n - \lfloor \frac{n}{3} \rfloor) \quad (18)$$

Based on (17) and (18), we have $pd(B_{D(K_n)}) = (n - \lfloor \frac{n}{3} \rfloor)$. The complete proof.

Figure 3 shows that the minimum resolving partition for $B_D(K_4)$ is 4, whereas Figure 4 for $B_{D(K_8)}$ is 6.

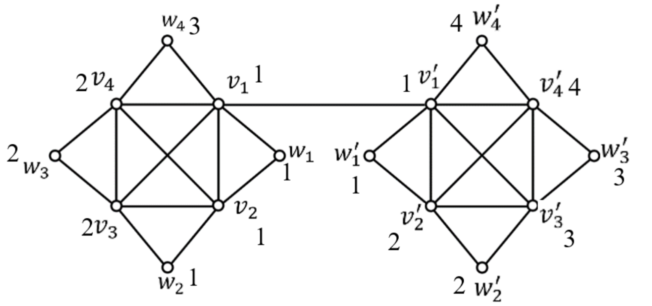

Figure 3. Graph $B_D(K_4)$ with Minimum Resolving Partition

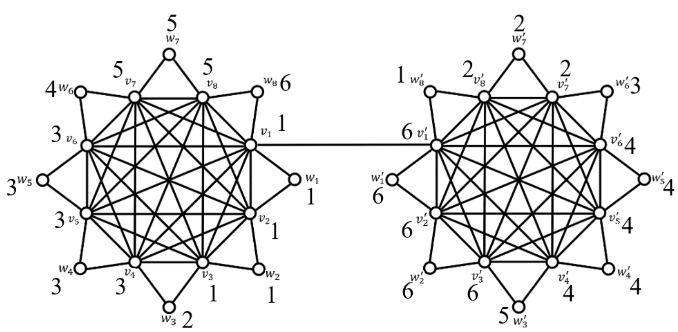

Figure 4. Graph $B_{D(K_8)}$ with Minimum Resolving Partition

4. CONCLUSION

Based on the above reasoning, the conclusion for this research is partition dimension of Daisy graph $pd(D(K_n))$ is 3 for $n = 3$, and $(n - \lfloor \frac{n}{3} \rfloor)$ for $n > 3$. Partition dimension of Barbell Daisy graph $pd(B_{D(K_n)})$ is 4 for $n = 3, 4$ and $(n - \lfloor \frac{n}{3} \rfloor)$ for $n > 4$.

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