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Research Paper

Transmission-Based Energies of Prime Coprime Graph for Integers Modulo Group

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Abstract

Graphs are an excellent instrument that provides an algebraic structure for visualizing and interpreting molecule structures and characteristics. As a result, the problem statement arises regarding how we can interpret graphs with eigenvalues concerning their corresponding matrices. Such questions can be answered by studying spectral graph theory. This research focuses on graphs whose vertex sets are group elements in which the structure of $\mathbb{Z}_n$ groups and the definition of a prime coprime graph serve as the foundation for the graph building used in this study. The matrix construction of the graph is based on transmission-based matrices including Weiner-Hosoya and distance signless Laplacian matrices. Research methods include investigating the transmission properties and formulation of the characteristic equation using block matrices. The results obtained are a comprehensive analysis of eigenvalues, spectrum, and spectral radius leading to the prime coprime graph energy for $\mathbb{Z}_n$ groups corresponding to both matrices.

Keywords

Wiener-Hosoya Matrix, Distance Signless Laplacian Matrix, Prime Coprime Graph, Integers Modulo Group, Energy of a Graph

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1. INTRODUCTION

Chemical graph theory is a field within science and mathematics focusing on chemical-based graphs representing molecular structures and interactions. These graphs provide a mathematical approach to understanding molecular activity that can be widely used in cheminformatics (Masmali et al., 2024). A chemical graph simulates a chemical system by depicting the interactions between its constituent atoms, bonds, and molecular groups. The chemical compound structure is a graph viewed geometrically, in which vertices denote atoms and edges are covalent bonds (Vijayalakshmi et al., 2023). Additionally, the boiling point of compounds can be explained by the theory of chemical graphs as seen in (Ali et al., 2025). In this context, various graphs have been evolved to observe structure-property and (quantitative) structure-activity relationships (QSAR and QSPR) (Veeragoudar et al., 2022).

One of the parameters that is becoming more popular in chemical graph theory is the spectrum of a graph. In addition, the potential applications of simple graphs represented by weighted symmetry matrices in modeling molecular interactions have been studied (Diaz-Diaz and Estrada, 2025). These advancements show how beneficial graph theory can be in chemistry, especially in explaining molecular stability, reactivity, and electronic properties. Some thermodynamic properties using graph energy of benzenoid hydrocarbons have been presented by (Sarkar et al., 2024). By documenting these interactions, an essential idea in relating adjacency matrices to graph theory has bridged the gap between algebraic representations and molecular structure.

First, the adjacency matrix represents the relationship between the graph and algebra. Gutman first defined graph energy in 1978 and was based on the eigenvalues of this matrix. Several authors have documented their findings on graph energy, such as Sombor energy (Romdhini and Nawawi, 2024) and the energy of picture fuzzy graphs (Shi et al., 2022). Moreover, several applications of graph energy have been demonstrated in satellite communication (Akram and Naz, 2018), protein sequence analysis and comparison (Sun et al., 2016), and molecular descriptors (Basak, 2024). Other applications for alkane identification (Shetty and Bhat, 2024) and polycyclic aromatic hydrocarbon chemical properties (Kumar et al., 2023) have also been reported.

The transmission concept of a graph is based on the distance between two vertices. The mathematical properties of the transmission-based graph have been extensively studied including a survey on transmission-based graphs (Sharafdini and Azadimotlagh, 2024), the spectral of complementary distance signless Laplacian matrix of graphs (Ramane et al., 2019), Wiener-Hosoya matrix (Ibrahim et al., 2021; Romdhini and Nawawi, 2024), and distance signless Laplacian matrix (Indulal et al., 2008; Khan et al., 2025). These properties can be associated with the graph defined on groups, including the coprime graph (Sehgal et al., 2021), the cubic power graph (Rana et al., 2024), and the prime coprime graph (Adhikari and Banerjee, 2022). The connection between vertices in the prime coprime graph is considered the interaction between group elements. In this graph, we can determine the degree and distance, and ultimately obtain the transmission properties.

In this research, we explore the transmission-based energies of the prime coprime graph focusing on group integer modulo n. The transmission-based matrices are Wiener-Hosoya and distance signless Laplacian matrices. On the other hand, we consider three cases when n is a prime number, n is the power of a prime, and in the last case, n is the product of two primes.

This paper is managed as follows. The basic theory and the notations are stated in the preliminaries section. In Sections 3 and 4, we present the results starting with the method to assign the characteristic formula of a matrix which is beneficial to solving the energy in the subsequent subsection. Transmission properties, investigation of both matrices for of prime coprime graph for $\mathbb{Z}_n$ group, and observation of their spectrum, and energy are also presented in the result section. The last section shows a summary of the conclusion.

Now we provide the basic definition and notation of spectral graph theory. Let us begin by mentioning the notation and its definition in the following table.

Table 1. Notation and its Definition

Symbol	Definition
G	group
Γ_G	prime coprime graph of G
$\lvert u\rvert$	order of u in G
$\mathbb{Z}_n$	group of integers modulo n
$\Gamma_{\mathbb{Z}_n}$	prime coprime graph of $\mathbb{Z}_n$
$deg(u)$	degree of vertex u
d_{uv}	distance between vertices u and v
τ_u	transmission of vertex u
$WH(\Gamma_{\mathbb{Z}_n})$	Wiener-Hosoya matrix of $\Gamma_{\mathbb{Z}_n}$
$D(\Gamma_{\mathbb{Z}_n})$	distance matrix of $\Gamma_{\mathbb{Z}_n}$
$T(\Gamma_{\mathbb{Z}_n})$	transmission matrix of $\Gamma_{\mathbb{Z}_n}$
$DSL(\Gamma_{\mathbb{Z}_n})$	distance signless Laplacian matrix of $\Gamma_{\mathbb{Z}_n}$
λ_i	eigenvalues of the matrix
$E_{WH}(\Gamma_{\mathbb{Z}_n})$	Wiener-Hosoya energy of $\Gamma_{\mathbb{Z}_n}$
$E_{DSL}(\Gamma_{\mathbb{Z}_n})$	distance signless Laplacian energy of $\Gamma_{\mathbb{Z}_n}$
$Spec(\Gamma_{\mathbb{Z}_n})$	spectrum of $\Gamma_{\mathbb{Z}_n}$
$\rho_{WH}(\Gamma_{\mathbb{Z}_n})$	WH-spectral radius of $\Gamma_{\mathbb{Z}_n}$
$\rho_{DSL}(\Gamma_{\mathbb{Z}_n})$	DSL-spectral radius of $\Gamma_{\mathbb{Z}_n}$

The prime coprime graph of a finite group G is denoted by $\Gamma_{\mathbb{Z}_n}$, in which G is the vertex set, and $u \neq v \in G$ are connected whenever $gcd(|u|, |v|)$ is equal to 1 or a prime number (Adhikari and Banerjee, 2022). It should be noted that $\Gamma_{\mathbb{Z}_n}$ is a simple graph. In further discussion, the prime coprime graph for $\mathbb{Z}_n$ is denoted by $\Gamma_{\mathbb{Z}_n}$. Graph $\Gamma_{\mathbb{Z}_n}$ corresponds to $WH(\Gamma_{\mathbb{Z}_n})$ and $DSL(\Gamma_{\mathbb{Z}_n})$. In the following definitions, both matrices are defined. Now let $deg(u)$ be the degree of vertex u. For vertices u and v, d_{uv} is the distance between them. Meanwhile, the sum of d_{uv}, for all $v \in \Gamma_{\mathbb{Z}_n}$, is the transmission of u and is denoted by τ_u (Ibrahim et al., 2021).

Definition 1.1. (Ibrahim et al., 2021) The WH-matrix of $\Gamma_{\mathbb{Z}_n}$, $WH(\Gamma_{\mathbb{Z}_n}) = [w_{uv}]$ in which the entries are

$$w_{uv} = \begin{cases} \dfrac{\tau_u}{2\cdot\deg(u)} + \dfrac{\tau_v}{2\cdot\deg(v)}, & \text{if } u \text{ and } v \text{ are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 1.2. (Indulal et al., 2008) The distance (D) matrix of $\Gamma_{\mathbb{Z}_n}$, $D(\Gamma_{\mathbb{Z}_n})$, is a square matrix in which entries are d_{uv} for $u \neq v$, and zero if $u = v$.

Definition 1.3. (Pirzada and Haq, 2023) The DSL-matrix of $\Gamma_{\mathbb{Z}_n}$ is $DSL(\Gamma_{\mathbb{Z}_n}) = D(\Gamma_{\mathbb{Z}_n}) + T(\Gamma_{\mathbb{Z}_n})$, where $T(\Gamma_{\mathbb{Z}_n})$ is an $n \times n$ diagonal matrix of the transmission of every vertex.

The Wiener-Hosoya energy formulation is $E_{WH}(\Gamma_{\mathbb{Z}_n}) = \Sigma_{i=1}^{m} |\lambda_i|$ (Gutman, 1978), in which $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the eigenvalues of $WH(\Gamma_{\mathbb{Z}_n})$. Additionally, the spectrum of $\Gamma_{\mathbb{Z}_n}$, is $Spec\,(\Gamma_{\mathbb{Z}_n})$, $\left\{\lambda_1^{k_1}, \lambda_2^{k_2}, \ldots, \lambda_m^{k_m}\right\}$ associated with $WH(\Gamma_{\mathbb{Z}_n})$ with respective multiplicities $k_1, k_2, \ldots, k_m$. The spectral radius of $\Gamma_{\mathbb{Z}_n}$ is $\rho_{WH}(\Gamma_{\mathbb{Z}_n}) = max\{|\lambda| : \lambda \in Spec(\Gamma_{\mathbb{Z}_n})\}$ (Ibrahim et al., 2021). The above notations also apply for $DSL(\Gamma_{\mathbb{Z}_n})$.

2. EXPERIMENTAL SECTION

2.1 Methods

This part formulates the characteristic polynomial of a particular matrix form. This finding is beneficial to solving the spectrum of $\Gamma_{\mathbb{Z}_n}$ in the subsequent section.

Theorem 2.1 If a, b, c, d, e are real numbers (not necessarily distinct) with m, n are natural and $m \leq n$, then the characteristic polynomial of

$$M = \begin{bmatrix} (a-b)I_m + bJ_m & cJ_{m\times(n-m)} \\ dJ_{(n-m)\times m} & (e-f)I_{n-m} + fJ_{n-m} \end{bmatrix}_{n\times n}$$

can be formulated as

$$\begin{aligned} P_M(\lambda) = {} & (\lambda - a + b)^{m-1}(\lambda - e + f)^{n-m-1} \\ & \times \Big(\lambda^2 - (a + e + b(m-1) + (n-m-1)f)\lambda \\ & + cdm(m-n) + (a + b(m-1)) \\ & (e + (n-m-1)f)\Big) \end{aligned}$$

Proof. For real numbers a, b, c, d, e, the characteristic formula of M is

$$P_M(\lambda) = \begin{vmatrix} (\lambda - a + b)I_m - bJ_m & -cJ_{m\times(n-m)} \\ -dJ_{(n-m)\times m} & (\lambda - e + f)I_{n-m} - fJ_{n-m} \end{vmatrix}$$

We first employ row and column operations on $P_M(\lambda)$ and denote R_i and C_i as the i-th row and column, respectively. By replacing R_{1+i} with $R_{1+i} - R_1$ for $1 \leq i \leq m-1$, we get

$$P_M(\lambda)=\begin{vmatrix} \lambda-a & -bJ_{1\times(m-1)} & -c & -cJ_{1\times(n-m-1)} \\ -(\lambda-a+b)J_{(m-1)\times 1} & (\lambda-a+b)I_{m-1} & 0_{(m-1)\times 1} & 0_{(m-1)\times(n-m-1)} \\ -d & -dJ_{1\times(m-1)} & \lambda-e & -fJ_{1\times(n-m-1)} \\ -dJ_{(n-m-1)\times 1} & -dJ_{(n-m-1)\times(m-1)} & -fJ_{(n-m-1)\times 1} & (\lambda-e)I_{n-m-1} \end{vmatrix}$$

It follows by replacing R_{m+1+i} with $R_{m+1+i}-R_{m+1}$ for $1 \le i \le n-m-1$, then we have $P_M(\lambda)$ as follows:

$$P_M(\lambda)=\begin{vmatrix} \lambda-a & -bJ_{1\times(m-1)} & -c & -cJ_{1\times(n-m-1)} \\ -(\lambda-a+b)J_{(m-1)\times 1} & (\lambda-a+b)I_{m-1} & 0_{(m-1)\times 1} & 0_{(m-1)\times(n-m-1)} \\ -d & -dJ_{1\times(m-1)} & \lambda-e & -fJ_{1\times(n-m-1)} \\ 0_{(n-m-1)\times 1} & 0_{(n-m-1)\times(m-1)} & (-\lambda+e-f)J_{(n-m-1)\times 1} & (\lambda-e+f)I_{n-m-1} \end{vmatrix}$$

The next step is to replace C_1 with $C_1+C_2+C_3+\ldots+C_m$, replace C_{m+1} with $C_{m+1}=C_{m+1}+C_{m+2}+\ldots+C_n$, and replace R_{m+1} with $R_{m+1}-R_1$ followed by replacing C_1 with $C_1-\left(\frac{(\lambda-a-b(m-1))d+bdm}{cd(n-m)+(\lambda-e-(n-m-1)f)b}\right)C_{m+1}$, then we have

$$\begin{aligned} P_M(\lambda) = {} & (\lambda-a+b)^{m-1}(\lambda-e+f)^{n-m-1} \\ & \times\Big(\lambda^2-\big(a+e+b(m-1) \\ & +(n-m-1)f\big)\lambda \\ & +cdm(m-n)+(a+b(m-1)) \\ & \times(e+(n-m-1)f)\Big) \end{aligned}$$

3. RESULTS AND DISCUSSION

3.1 Transmission of Every Vertex in Prime Coprime Graph for $\mathbb{Z}_n$

In this section, we obtain the transmission of every vertex in $\Gamma_{\mathbb{Z}_n}$ for $n=p$, p^k, or pq, where p,q are prime numbers and the natural number k.

Theorem 3.1 For $n=p$ and p prime number, the transmission of u in $\mathbb{Z}_n$ is

$$\tau_u = p-1.$$

Proof. Let $n=p$ and p a prime number. Each vertex in $\Gamma_{\mathbb{Z}_n}$ is connected to all other vertices in $\Gamma_{\mathbb{Z}_n}$. Hence, the distance between two distinct vertices is one. Consequently, the transmission of u is

$$\tau_u=(p-1)\cdot 1=p-1,$$

and the proof is completed.

Theorem 3.2 The transmission of u in $\Gamma_{\mathbb{Z}_n}$, where $n=p^k$, p prime number, and an integer $k\ge 2$, is

$$\tau_u=\begin{cases} p^k-1 & , \text{ for } u\in V_1 \\ 2p^k-p-2 & , \text{ for } u\in V_2, \end{cases}$$

$$\begin{aligned} &\text{where } V=\{0,1,2,\ldots,p^{k-1}\}, \\ &V_1=\{0p^{k-1},1p^{k-1},2p^{k-1},\ldots,(p-1)p^{k-1}\}, \\ &\qquad\qquad\qquad\qquad \text{and } V_2=V\setminus V_1 \end{aligned}$$

Proof. Let $n=p^k$, $V=\{0,1,2,\ldots,p^{k-1}\}$, $V_1=\{0p^{k-1},1p^{k-1},2p^{k-1},\ldots,(p-1)p^{k-1}\}$, and $V_2=V\setminus V_1$. Consider the first case: for every $u\in V_1$, note that u is always adjacent to every $v\in V$. Then $d_{uv}=1$ for all $v\in V$. Since $|V|=p^k$, consequently, the transmission of u is p^k-1. Meanwhile, for every $u\in V_2$ and $v\in V_2$, the distance between both vertices is 2. According to $|V_2|=|V|-|V_1|=p^k-p$, then the transmission of u is

$$\tau_u=(p\cdot 1)+((p^k-p-1)\cdot 2)=2p^k-p-2$$

Theorem 3.3 The transmission of u in $\Gamma_{\mathbb{Z}_n}$, where $n=pq$, where p,q are prime numbers, is

$$\tau_u=\begin{cases} pq-1 & , \text{ for } u\in H \\ 2pq-(p+q-1) & , \text{ for } u\in \mathbb{Z}_n\backslash H \end{cases}$$

where $H=\{p,2p,3p,\ldots,qp\}\cup\{q,2q,3q,\ldots,pq\}$. Proof. Let $H=\{p,2p,3p,\ldots,qp\}\cup\{q,2q,3q,\ldots,pq\}$ with $|H|=p+q-1$. We know that $u\in H$ is always adjacent to all members of $\mathbb{Z}_n$. Then for every $v\in\mathbb{Z}_n$ with $u\ne v$, the distance between u and v is 1. Since $n=pq$, then the transmission of u is $pq-1$. Meanwhile, for every $u,v\in\mathbb{Z}_n\backslash H$, the distance between both vertices is 2. According to $|\mathbb{Z}_n\backslash H|=|\mathbb{Z}_n|-|H|=pq-(p+q-1)$, consequently the transmission of u is

$$\tau_u=((p+q-1)\cdot 1)+((pq-(p+q-1))\cdot 2)=2pq-(p+q-1)$$

3.2 Wiener-Hosoya Energy of Prime Coprime Graph for $\mathbb{Z}_n$

This section focuses on the formulation of the spectrum, spectral radius, and energy of $\mathbb{Z}_n$ based on the Wiener-Hosoya matrix. We divide it into three cases when $n=p$, p^k, and pq, for prime numbers p,q and the natural number k.

Theorem 3.4 For $n=p$ and p prime number, the Wiener-Hosoya energy of $\Gamma_{\mathbb{Z}_n}$ is

$$E_{WH}(\Gamma_{\mathbb{Z}_n})=2(p-1)$$

Proof. For $n=p$ and a prime number p, $\Gamma_{\mathbb{Z}_n}$ is a complete graph. We know that $deg(u)=p-1$, meanwhile, Theorem ?? implies $\tau_u=p-1$, for every vertex u in $\Gamma_{\mathbb{Z}_n}$. Hence, the non-diagonal entries of $WH(\Gamma_{\mathbb{Z}_n})$ is $\frac{p-1}{2(p-1)}+\frac{p-1}{2(p-1)}=1$. Therefore, the construction of $WH(\Gamma_{\mathbb{Z}_n})$ is similar to the adjacency matrix of a complete graph on p vertices. It immediately follows that $E_{WH}(\Gamma_{\mathbb{Z}_n})$ is $2(p-1)$.

Theorem 3.5 For $n=p^k$, where p prime number, and an integer $k\ge 2$, the Wiener-Hosoya energy of $\Gamma_{\mathbb{Z}_n}$ is

$$\begin{aligned} E_{WH}(\Gamma_{\mathbb{Z}_n})={} & p-1+ \\ & \sqrt{(p-1)^2+4(p^{k-1})(2p^k-p-2)(p^{k-1}-1)} \end{aligned}$$

Proof. Based on Theorem 3.1, for $u\in V_1$, we have $\tau_u=p^{k-1}$ and $deg(u)=p^{k-1}$. Consequently, for $u,v\in V_1$, the entries of $WH(\Gamma_{\mathbb{Z}_n})$ are $\frac{p^k-1}{2(p^k-1)}+\frac{p^k-1}{2(p^k-1)}=1$. If $v\in V_2$, we know that $\tau_v=2p^k-p-2$ conforming Theorem 3.1, and $deg(v)=p$.

Hence, the entries of $WH(\Gamma_{\mathbb{Z}_n})$ for $u \in V_1$ and $v \in V_2$ or vice versa are $\frac{p^k-1}{2(p^k-1)} + \frac{2p^k-p-2}{2p} = \frac{p^k-1}{p}$ and for $u, v \in V_2$ the entries are $\frac{2p^k-p-2}{2p} + \frac{2p^k-p-2}{2p} = \frac{2p^k-p-2}{p}$. Thus, the construction of $WH(\Gamma_{\mathbb{Z}_n})$ is

$$\begin{array}{c|ccccccccccccc|}
0 & 0 & 0 & 1 & \cdots & p-1 & p & p+1 & \cdots & p^2-1 & \cdots & p^{m-1}+1 & \cdots & p^m-1 \\
1 & \frac{2p^k-p-2}{p} & 0 & 0 & \cdots & 0 & \frac{p^k-1}{p} & 0 & \cdots & 0 & \cdots & \frac{p^k-1}{p} & \cdots & 0 \\
\vdots & \frac{2p^k-p-2}{p} & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p-1 & \frac{2p^k-p-2}{p} & 0 & 0 & \cdots & 0 & \frac{p^k-1}{p} & 0 & \cdots & 0 & \cdots & \frac{p^k-1}{p} & \cdots & 0 \\
p & 1 & \frac{p^k-1}{p} & \frac{p^k-1}{p} & \cdots & \frac{p^k-1}{p} & 0 & \frac{p^k-1}{p} & \cdots & \frac{p^k-1}{p} & \cdots & 0 & \cdots & \frac{p^k-1}{p} \\
p+1 & \frac{2p^k-p-2}{p} & 0 & 0 & \cdots & 0 & \frac{p^k-1}{p} & 0 & \cdots & 0 & \cdots & \frac{p^k-1}{p} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p^2-1 & \frac{2p^k-p-2}{p} & 0 & 0 & \cdots & 0 & \frac{p^k-1}{p} & 0 & \cdots & 0 & \cdots & \frac{p^k-1}{p} & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\
p^m-1 & \frac{2p^k-p-2}{p} & 0 & 0 & \cdots & 0 & \frac{p^k-1}{p} & 0 & \cdots & 0 & \cdots & \frac{p^k-1}{p} & \cdots & 0
\end{array}$$

By applying row and column operations $R_{p^i} \rightarrow R_i$ and followed by $C_{p^i} \rightarrow C_i$, then we have

$$WH(\Gamma_{\mathbb{Z}_n}) = \begin{bmatrix} (J-I)_p & \left(\frac{p^k-1}{p}\right) J_{p\times(n-p)} \\ \left(\frac{2p^k-p-2}{p}\right) J_{(n-p)\times p} & 0_{n-p} \end{bmatrix}$$

Based on Theorem 2.1 with $a = 0, b = 1, c = \frac{p^k-1}{p}, d = \frac{2p^k-p-2}{p}, e = 0, f = 0$, and $m = p$, we get the characteristic formula of $WH(\Gamma_{\mathbb{Z}_n})$ as

$$\begin{aligned} P_{WH(\Gamma_{\mathbb{Z}_n})}(\lambda) = {} & (\lambda+1)^{p-1}\lambda^{p^k-p-1} \\ & \times \Big(\lambda^2 - (p-1)\lambda \\ & - (p^k-1)(2p^k-p-2)(p^{k-1}-1)\Big) \end{aligned}$$

The roots of $P_{WH(\Gamma_{\mathbb{Z}_n})}(\lambda)$ are the eigenvalues of $WH(\Gamma_{\mathbb{Z}_n})$ which means $(\lambda+1)^{p-1} = 0$, $(\lambda)^{p^k-p-1} = 0$, and $\lambda^2 - (p-1)\lambda - (p^k-1)(2p^k-p-2)(p^{k-1}-1) = 0$. Thus we obtain $\lambda_1 = 0$ and $\lambda_2 = -1$ with multiplicity $p^k - p - 1$ and $p - 1$, respectively, and $\lambda_{3,4} = \frac{1}{2}\left(p - 1 \pm \sqrt{(p-1)^2 + 4(p^k-1)(2p^k-p-2)(p^{k-1}-1)}\right)$ of each multiplicity 1. This, the WH-spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$Spec(\Gamma_{\mathbb{Z}_n}) = \left\{ \left(\frac{1}{2}\left(p-1+\sqrt{\begin{array}{c}(p-1)^2+4(p^k-1)\\(2p^k-p-2)\\(p^{k-1}-1)\end{array}}\right)\right)^1, \right.$$
$$\left. \begin{array}{l} (-1)^{p-1}, \\ (0)^{p^k-p-1}, \\ \left(\frac{1}{2}\left(p-1-\sqrt{\begin{array}{c}(p-1)^2+4(p^k-1)\\(2p^k-p-2)(p^{k-1}-1)\end{array}}\right)\right)^1 \end{array} \right\}$$

For $i = 1, 2, 3, 4$, according to $Spec(\Gamma_{\mathbb{Z}_n})$, WH-spectral radius of $\Gamma_{\mathbb{Z}_n}$ can be determined as the maximum of $|\lambda_i|$,

$$\rho_{WH}(\Gamma_{\mathbb{Z}_n}) = \frac{1}{2}\left(p-1+\sqrt{\begin{array}{c}(p-1)^2+4(p^k-1)\\ \times(2p^k-p-2)(p^{k-1}-1)\end{array}}\right)$$

The Wiener-Hosoya energy of $\Gamma_{\mathbb{Z}_n}$ is

$$\begin{aligned} E_{WH}(\Gamma_{\mathbb{Z}_n}) = {} & (p^k-p-1)\,|0| + (p-1)\,|-1| + \Bigg|\frac{1}{2}(p-1\pm \\ & \sqrt{(p-1)^2+4(p^k-1)(2p^k-p-2)(p^{k-1}-1)}\Bigg| \\ = {} & p-1+ \\ & \sqrt{(p-1)^2+4(p^k-1)(2p^k-p-2)(p^{k-1}-1)} \end{aligned}$$

Theorem 3.6 For prime number p, q and $n = pq$, the Wiener-Hosoya energy of $\Gamma_{\mathbb{Z}_n}$ is

$$E_{WH}(\Gamma_{\mathbb{Z}_n}) = p+q-2+\sqrt{(p+q-2)^2 + \frac{\begin{array}{c}((pq-2)(p+q+2)-3)\cdot\\(p+q-1-2pq)\cdot\\(p+q-1-pq)\end{array}}{4(pq-1)(p+q-1)}}$$

Proof. By Based on Theorem 3.1, for $u \in H$, we have $\tau_u = pq - 1$ and $deg(u) = pq - 1$. Consequently, for $u, v \in H$, the entries of $WH(\Gamma_{\mathbb{Z}_n})$ are $\frac{pq-1}{2(pq-1)} + \frac{pq-1}{2(pq-1)} = 1$. If $v \in \mathbb{Z}_n \backslash H$, we know that $\tau_v = 2pq - (p + q - 1)$, conforming Theorem 3.2, and $deg(v) = p + q - 1$. Hence, the entries of $WH(\Gamma_{\mathbb{Z}_n})$ for $u \in H$ and $v \in \mathbb{Z}_n \backslash H$ or vice versa are $\frac{pq-1}{2(pq-1)} + \frac{2pq-(p+q-1)}{2(p+q-1)} = \frac{(pq-2)(p+q+2)-3}{4(pq-1)(p+q-1)}$ and for $u, v \in V_2$ the entries are $\frac{2pq-(p+q-1)}{2(p+q-1)} + \frac{2pq-(p+q-1)}{2(p+q-1)} = \frac{2pq-(p+q-1)}{p+q-1}$. Thus, the construction of $WH(\Gamma_{\mathbb{Z}_n})$ is

$$\begin{bmatrix} (J-I)_{p+q-1} & \begin{array}{c}\left(\frac{(pq-2)(p+q+2)-3}{4(pq-1)(p+q-1)}\right)\\ \cdot J_{(p+q-1)\times(n-(p+q-1))}\end{array} \\ \left(\frac{2pq-(p+q-1)}{p+q-1}\right) J_{(n-(p+q-1))\times(p+q-1)} & 0_{n-(p+q-1)} \end{bmatrix}$$

Based on Theorem 2.1 with $a = 0$, $b = 1$, $c = \frac{(pq-2)(p+q+2)-3}{4(pq-1)(p+q-1)}$, $d = \frac{2pq-(p+q-1)}{p+q-1}$, $e = 0$, $f = 0$, and $m = p + q - 1$, we get the characteristic formula of $WH(\Gamma_{\mathbb{Z}_n})$ as

$$\begin{aligned} & P_{WH(\Gamma_{\mathbb{Z}_n})}(\lambda) = (\lambda+1)^{p+q-2}(\lambda)^{pq-(p+q)} \\ & \Bigg(\lambda^2 - (p+q-2)\lambda - \\ & \frac{((pq-2)(p+q+2)-3)(p+q-1-2pq)(p+q-1-pq)}{4(pq-1)(p+q-1)}\Bigg) \end{aligned}$$

Considering $P_{WH(\Gamma_{\mathbb{Z}_n})}(\lambda) = 0$ implies $(\lambda+1)^{p+q-2} = 0$, $(\lambda)^{pq-(p+q)} = 0$, and $\lambda^2 - (p+q-2)\lambda - \frac{((pq-2)(p+q+2)-3)(p+q-1-2pq)(p+q-1-pq)}{4(pq-1)(p+q-1)} = 0$. Then we get $\lambda_1 = -1$ and $\lambda_2 = 0$ of multiplicities $p + q - 2$ and $pq - (p + q)$, respectively. We also get

$$\lambda_{3,4} = \frac{1}{2}\left(p+q-2 \pm \sqrt{(p+q-2)^2 + \frac{((pq-2)(p+q+2)-3)\cdot(p+q-1-2pq)(p+q-1-pq)}{4(pq-1)(p+q-1)}}\right)$$

of each multiplicity 1. Thus, the WH-spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$\rho_{WH}(\Gamma_{\mathbb{Z}_n}) = \frac{1}{2}\left(p+q-2+\sqrt{(p+q-2)^2 + \frac{((pq-2)(p+q+2)-3)\cdot(p+q-1-2pq)\cdot(p+q-1-pq)}{4(pq-1)(p+q-1)}}\right)$$

The Wiener-Hosoya energy of $\Gamma_{\mathbb{Z}_n}$ is

$$E_{WH}(\Gamma_{\mathbb{Z}_n}) = (pq-(p+q))\,|0| + (p+q-2)\,|-1| + \left|\frac{1}{2}\left(p+q-2 \pm \sqrt{(p+q-2)^2 + \frac{((pq-2)(p+q+2)-3)\cdot(p+q-1-2pq)\cdot(p+q-1-pq)}{4(pq-1)(p+q-1)}}\right)\right|$$

$$= p+q-2+\sqrt{(p+q-2)^2 + \frac{((pq-2)(p+q+2)-3)\cdot(p+q-1-2pq)\cdot(p+q-1-pq)}{4(pq-1)(p+q-1)}}$$

3.3 Distance Signless Laplacian Energy of Prime Coprime Graph for $\mathbb{Z}_n$

The other type of transmission-based matrix is the DSL-matrix. We discuss it in this section including the discussion of spectrum, spectral radius, and energy of $\Gamma_{\mathbb{Z}_n}$. We divide it into three cases when $n = p$, p^k, and pq, for prime numbers p, q, and natural number k.

Theorem 3.7 For $n = p$ and p prime number, the distance signless Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ is

$$E_{DSL}(\Gamma_{\mathbb{Z}_n}) = p(p-1).$$

Proof. For n is a prime number p, $\Gamma_{\mathbb{Z}_n}$ is a complete graph. We know that the distance of every two vertices is 1. From Theorem 3.1 we have $\tau_u = p-1$, for every vertex u in $\Gamma_{\mathbb{Z}_n}$. As a consequence of Definition 1, the diagonal entries of $DSL(\Gamma_{\mathbb{Z}_n})$ are $p-1$ and the non-diagonal entries are 1. Therefore, the construction of $DSL(\Gamma_{\mathbb{Z}_n})$ is

$$DSL(\Gamma_{\mathbb{Z}_n}) = \begin{bmatrix} p-1 & 1 & \dots & 1 \\ 1 & p-1 & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \dots & p-1 \end{bmatrix}$$

$$= [(p-2)I_n + J_n]$$

Accordingto Theorem 2.1 in which $a = p-1$, $b = 1$, $c = d = e = f = 0$, and $m = n = p$, then the characteristic formula of $DSL(\Gamma_{\mathbb{Z}_n})$ is

$$P_{DSL(\Gamma_{\mathbb{Z}_n})}(\lambda) = (\lambda - p + 2)^{p-1}\,(\lambda - 2(p-1))$$

It immediately follows that the DSL-energy of $\Gamma_{\mathbb{Z}_n}$ is $p(p-1)$.

Theorem 3.8 For $n = p^k$, where p prime number, and an integer $k \geq 2$, the DSL-energy of $\Gamma_{\mathbb{Z}_n}$ is

$$E_{DSL}(\Gamma_{\mathbb{Z}_n}) = 2p^k(p^k - p - 1) + p(p+1).$$

Proof. Based on Theorem 3.2, for $u \in V_1$, we have $\tau_u = p^k - 1$. The distance between u and v is one, for every $v \in V$. Meanwhile, If $v \in V_2$, we know that $\tau_v = 2p^k - p - 2$ and for every $u \in V_2$, the distance between both vertices is 2. It follows that the construction of $DSL(\Gamma_{\mathbb{Z}_n})$ is

$$\begin{array}{c} 0 \\ 1 \\ \\ \vdots \\ p-1 \\ p \\ p+1 \\ \vdots \\ p^2-1 \\ \vdots \\ p^m-1 \\ p^m-1+1 \\ \vdots \\ p^m-1 \end{array}
\begin{bmatrix}
0 & 1 & \cdots & \cdots & p-1 & p & p+1 & \cdots & p^2-1 & \cdots & p^{m-1}+1 & \cdots & p^m-1 \\
p^k-1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 2p^k-p-2 & \cdots & 2 & 1 & 2 & \cdots & 2 & \cdots & 1 & 2 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & \cdots & 2p^k-p-2 & 1 & 2 & \cdots & 2 & \cdots & 1 & 2 & \cdots & 2 \\
1 & 1 & \cdots & 1 & p^k-1 & 1 & \cdots & 1 & \cdots & 1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2 & 1 & 2p^k-p-2 & \cdots & 2 & \cdots & 1 & 2 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & \cdots & 2 & 1 & 2 & \cdots & 2p^k-p-2 & \cdots & 1 & 2 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 & \cdots & p^2-1 & 1 & \cdots & 1 \\
1 & 2 & \cdots & 2 & 1 & 2 & \cdots & 2 & \cdots & 1 & 2p^k-p-2 & \cdots & 2 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 2 & \cdots & 2 & 1 & 2 & \cdots & 2 & \cdots & 1 & 2 & \cdots & 2p^k-p-2
\end{bmatrix}$$

By applying row and column operations $R_{p^i} \rightarrow R_i$ and followed by $C_{p^i} \rightarrow C_i$, then we have

$$DSL(\Gamma_{\mathbb{Z}_n}) = \begin{bmatrix} (p^k-2)I_p + J_p & J_{p\times(n-p)} \\ J_{(n-p)\times p} & (2p^k-p-4)I_{n-p} + 2J_{n-p} \end{bmatrix}$$

Based on Theorem 2.1 with $a = p^k - 1$, $b = c = d = 1$, $e = 2p^k - p - 2$, $f = 2$, and $m = p$, we get the characteristic formula of $DSL(\Gamma_{\mathbb{Z}_n})$ as

$$\begin{aligned} P_{DSL(\Gamma_{\mathbb{Z}_n})}(\lambda) = & (\lambda - (p^k-2))^{p-1} \\ & (\lambda - (2p^k-p-4))^{p^k-p-1} \\ & \Big(\lambda^2 - (5p^k - 2p - 6)\lambda + \\ & 4p^k(p^k-3) - 2(p^2-p-4)\Big) \end{aligned}$$

Then we get $\lambda_1 = 2p^k - p - 4$ (multiplicity $p^k - p - 1$), $\lambda_2 = p^k - 2$ (multiplicity $p - 1$), $\lambda_{3,4} = \frac{1}{2}\Bigg(5p^k - 2p - 6 \pm$

$$\sqrt{\begin{matrix}(5p^k-2p-6)^2 - 16p^k(p^k-3) \\ +8(p^2-p-4)\end{matrix}}\Bigg)$$

of each multiplicity 1. Thus, the DSL-spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$\mathrm{Spec}(\Gamma_{\mathbb{Z}_n}) = \left\{\left(\frac{1}{2}\left(5p^k - 2p - 6 + \sqrt{\begin{matrix}(5p^k-2p-6)^2 \\ -16p^k(p^k-3) \\ +8(p^2-p-4)\end{matrix}}\right)\right)^1,\right.$$

$$\left(2p^k - p - 4\right)^{p^k-p-1}, \quad \left(p^k - 2\right)^{p-1},$$

$$\left(\frac{1}{2}\left(5p^k - 2p - 6 - \sqrt{\begin{matrix}(5p^k - 2p - 6)^2 \\ - 16p^k(p^k - 3) \\ + 8(p^2 - p - 4)\end{matrix}}\right)\right)^1 \}$$

By $Spec(\Gamma_{\mathbb{Z}_n})$, the optimum of $|\lambda_i|$ where $i = 1, 2, 3, 4$, is

$$\rho_{DSL}(\Gamma_{\mathbb{Z}_n}) = \frac{1}{2}\left(5p^k - 2p - 6 + \sqrt{\begin{matrix}(5p^k - 2p - 6)^2 \\ - 16p^k(p^k - 3) \\ + 8(p^2 - p - 4)\end{matrix}}\right)$$

The DSL-energy of $\Gamma_{\mathbb{Z}_n}$ is

$$\begin{aligned} E_{DSL}(\Gamma_{\mathbb{Z}_n}) &= (p^k - p - 1)\,|0| + (p - 1)\left|p^k - 2\right| + \\ &\quad \left|\frac{1}{2}\left(5p^k - 2p - 6 \pm \sqrt{\begin{matrix}(5p^k - 2p - 6)^2 - 16p^k(p^k - 3) \\ +8(p^2 - p - 4)\end{matrix}}\right)\right| \\ &= 2p^k(p^k - p - 1) + p(p + 1) \end{aligned}$$

Theorem 3.9 For $n = pq$, where p, q are prime numbers, the distance signless Laplacian energy of $\Gamma_{\mathbb{Z}_n}$ is

$$E_{DSL}(\Gamma_{\mathbb{Z}_n}) = pq(2pq+3)-(p+q)(pq+4)+(p+q+1)(p+q-pq)+2.$$

Based on Theorem 3.1, for $u \in H$, we have $\tau_u = pq - 1$ and the distance between u and any other vertices in $\Gamma_{\mathbb{Z}_n}$ is one. If $v \in \mathbb{Z}_n \backslash H$, we know that $\tau_v = 2pq-(p+q-1)$, conforming Theorem 3.2, and $d_{uv} = 2$, for $v \in \mathbb{Z}_n \backslash H$. Hence, the construction of $DSL(\Gamma_{\mathbb{Z}_n})$ is

$$\begin{bmatrix} (pq - 2)I_{p+q-1} + J_{p+q-1} \\ J_{p\times(n-(p+q-1))} \\ J_{(n-(p+q-1))\times p} \\ (2pq - (p + q + 1))I_{pq-(p+q-1)} + 2J_{pq-(p+q-1)} \end{bmatrix}$$

Based on Theorem 2.1 with $a = pq - 1$, $b = c = d = 1$, $e = 2pq - (p + q - 1)$, $f = 2$ and $m = p + q - 1$, we get the characteristic formula of $DSL(\Gamma_{\mathbb{Z}_n})$ as

$$\begin{aligned} P_{DSL(\Gamma_{\mathbb{Z}_n})}(\lambda) = (\lambda - pq + 2)^{p+q-2} \\ (\lambda - 2pq + p + q + 1)^{pq-(p+q)} \\ (\lambda^2 - (5pq - 2(p + q + 1))\lambda \\ + (p + q - 1)(p + q - 1 - pq) \\ + (pq + p + q - 3)(4pq - 3(p + q) + 1)) \end{aligned}$$

Then we get $\lambda_1 = 2pq-(p+q+1)$ of multiplicity $pq-(p+q)$, $\lambda_2 = pq - 2$ of multiplicity $p + q - 2$, and

$$\begin{aligned} \lambda_{3,4} =& \frac{1}{2}\,(5pq - 2(p + q + 1)\pm \\ & \sqrt{\left(5pq - 2(p + q + 1))^2 - 4\left(m(m - pq)+ \right.\right.} \\ & \quad (pq + p + q - 3)(4pq - 3(p + q) + 1))\Big) \end{aligned}$$

of each multiplicity 1. Thus, the DSL-spectrum of $\Gamma_{\mathbb{Z}_n}$ is

$$\begin{aligned} &\left\{\left(\frac{1}{2}\,(5pq - 2(p + q + 1)\pm \right.\right. \\ &\sqrt{(5pq - 2(p + q + 1))^2 - 4\left(m(m - pq) + \right.} \\ &\quad \left.\left.(pq + p + q - 3)(4pq - 3(p + q) + 1)\right)\right)^1, \\ &(2pq - (p + q + 1))^{pq-(p+q)}, (pq - 2)^{p+q-2}, \\ &\left(\frac{1}{2}\,(5pq - 2(p + q + 1)\pm \right. \\ &\sqrt{(5pq - 2(p + q + 1))^2 - 4\left(m(m - pq) + \right.} \\ &\quad \left.\left.(pq + p + q - 3)(4pq - 3(p + q) + 1)\right)\right)^1, \} \end{aligned}$$

According to $Spec(\Gamma_{\mathbb{Z}_n})$, the DSL-spectral radius of $\Gamma_{\mathbb{Z}_n}$ is

$$\begin{aligned} \rho_{DSL}(\Gamma_{\mathbb{Z}_n}) =& \frac{1}{2}\Big(5pq - 2(p + q + 1) \pm \\ & \sqrt{(5pq - 2(p + q + 1))^2 - 4\Big(m(m - pq)} \\ & + (pq + p + q - 3)(4pq - 3(p + q) + 1)\Big)\Big) \end{aligned}$$

The DSL-energy of $\Gamma_{\mathbb{Z}_n}$ is

$$\begin{aligned} E_{DSL}(\Gamma_{\mathbb{Z}_n}) =& (pq - (p + q))|2pq - (p + q + 1)| + \\ & (p + q - 2)|pq - 2| + \Big|\frac{1}{2}\Big(5pq - 2 \\ & (p + q + 1)\pm \\ & \sqrt{(5pq - 2(p + q + 1))^2 - 4\Big(m(m - pq)} \\ & + (pq + p + q - 3)(4pq - 3(p + q) + 1)\Big)\Big| \\ =& pq(2pq + 3) - (p + q)(pq + 4)+ \\ & (p + q + 1)(p + q - pq) + 2 \end{aligned}$$

4. CONCLUSIONS

In this research, the formula for the energy of $\Gamma_{\mathbb{Z}_n}$ corresponding to the WH and DSL-matrices are provided. The findings have potential applications in computational chemistry, particularly in the areas of physical characteristics, chemical reactivity, or biological action. Moreover, it is possible to generalize the energy of a fuzzy graph based on transmission matrices, which can contribute to the development of decision-making theory.

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