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Research Paper

Commuting and Centralizing Maps on Modules

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Abstract
A ring is a mathematical structure composed of a set with two binary operations that follow certain axioms. One important function within a ring is the centralizing and commuting mapping, which has been extensively studied in recent decades. Commuting mappings are a special case of centralizing mappings. A module is a generalization of a ring. In this paper, we extend the concept of commuting mappings from ring to module structures. However, defining commuting mappings in modules presents a challenge, as multiplication is required for their definition, yet modules do not have this operation. Additionally, constructing nonzero centralizing and commuting mappings on modules is a nontrivial task. To address these challenges, we employ the concept of idealization as a framework for defining commuting mappings in modules. We also propose a method for constructing nonzero commuting mappings on modules by leveraging existing commuting mappings in rings. Specifically, if α is a commuting mapping on a ring T, then a corresponding commuting mapping α' can be defined on the module by utilizing α. Moreover, we establish that the finite sum of commuting mappings is also a commuting mapping and that a linear combination of commuting mappings is also a commuting mapping under certain conditions.

Keywords
Commuting, Centralizing, Module, Ring, Idealization

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1. INTRODUCTION

A ring is a mathematical structure composed of a set with two binary opera- tions that follow certain axioms. Rings play a fundamental role in various fields,including physics, chemistry, economics, finance, coding theory, and cryptography. Among the important functions in ring theory is the commuting mapping, which has been extensively studied in recent decades. Commuting mappings are a special case of centralizing mappings, and both concepts have seen significant development in algebra, particularly in areas such as commuting derivations, additive commuting mappings, and their applications in Lie theory and functional analysis. Moreover, the concept of commuting also finds meaningful applica- tions in graph theory (Romdhini and Nawawi, 2025), demonstrating the broad relevance of this idea across mathematical domains.

Commuting mappings play a crucial role in algebraic structures and have been the subject of extensive research. The study of these mappings has grown significantly, leading to numerous advancements. Brešar (2004) developed a comprehensive theory of commuting mappings in rings and explored their applications in Lie theory. Further contributions include Khan (2019), who studied commuting and centralizing involution mappings related to derivations in rings, and Hoque and Paul (2013), who examined commuting mappings in prime Gamma rings, identifying conditions for commutativity. More recently, Mahmood (2022) introduced a generalized concept of commuting mappings for prime and semiprimary rings, while Ebrahimi and Talebi (2015) investigated k- commuting structures in algebraic extension modules. Research on alternative ring structures has also gained attention, with Ferreira and Kaygorodov (2020) studying commuting mappings in alternative rings, followed by Ferreira et al. (2024) extending this work to alternative division rings. Moreover, De Filippis et al. (2024) extend the concept of derivation in prime rings.

While much of the existing research has focused on commuting mappings in rings, a module is a more general structure that extends the concept of rings. In this paper, we aim to extend commuting mappings from ring structures to module structures. However, this presents a fundamental challenge: commuting mappings typically require a multiplication operation, which does not naturally exist in modules. Additionally, constructing nonzero centralizing and commuting mappings on modules is nontrivial.

A potential solution to this challenge is the concept of idealization, which extends a module N over a ring T to form $T(+)N$. Originally introduced by Anderson and Winders (2009), idealization defines multiplication operations between elements of the extended structure. Subsequent research has explored its various properties, including Loan (2017), who studied the canonical module structure, and Tran (2018), who examined the primary decomposition of homogeneous ideals in idealized modules. Batanieh and Anderson (2014) also analyzed polynomial identities in these modules. Inspired by these developments, we use idealization as a framework to define centralizing and commuting mappings in module structures, effectively generalizing the definition of commutation from rings to modules. This approach helps overcome the absence of a multiplication operation in modules, making it a promising method for extending commuting mappings.

Another significant aspect of commuting mappings is derivation, a specialized type of mapping that has been widely studied in ring and module structures. Research on derivations in rings includes works by Güven (2008), Gölbaşı and Koç (2011), Ali et al. (2014), Ali and Alhazmi (2017), Mamouni and Tamekkante (2021), Aishwarya et al. (2023), Benkovič (2024), and Thomas et al. (2024). The concept of derivation has also been explored in module structures, with contributions from Bracic (2001), Patra and Gurjar (2022), Retert (2006), Chen and Wang (2015), Maubach (2003), and Pogudin (2019). More recently, Fitriani et al. (2025) introduced f-derivation on polynomial modules. In this paper, we employ idealization to construct nonzero commuting mappings on modules and polynomial modules by utilizing existing commuting mappings in rings.

Previous research has primarily focused on centralizing and commuting mappings in the framework of rings and Lie algebras, with idealization being studied mainly within the framework of commutative rings. Although, the extension of idealization to arbitrary rings and its use in defining centralizing and commuting mappings in modules remain unexplored. This paper aims to fill this gap by introducing a general framework for commuting mappings on modules $T(+)N$. Specifically, we extend the concept of centralizing and commuting mappings from rings to modules, develop a method to construct new commuting mappings on modules using existing commuting mappings in rings, investigate the properties and characterizations of centralizing and commuting mappings on $T(+)N$ over arbitrary rings, demonstrate how a commuting mapping α on a ring T can be broadened to define a commuting mapping α' on the module $T(+)N$. Furthermore, we show that the finite sum of commuting mappings is also a commuting mapping, and under certain conditions, a linear combination of commuting mappings also retains this property.

By extending the notion of commuting mappings to module structures, our findings contribute to a deeper understanding of algebraic structures and their applications, particularly in functional analysis and related fields.

2. EXPERIMENTAL DESIGN

In this study, we employ a theoretical and algebraic approach to extend the concept of commuting mappings from rings to module structures. Our expe- rimental design consists of the following methodological framework. We begin by establishing the foundational mathematical concepts necessary for our study. These include definitions of commuting and centralizing mappings in ring theory, properties and axioms of modules over arbitrary rings, the idealization technique $T(+)N$ as introduced by Anderson and Winders (2009), providing a structured approach to incorporating module elements within a ring-like framework. In the next step, we develop a systematic method to extend commuting mappings from rings to modules by utilizing idealization. The steps include:
1. constructing the idealization structure $T(+)N$ and defining its operations;
2. establishing conditions under which a commuting mapping α on a ring T induces a commuting mapping α' on $T(+)N$;
3. verifying algebraic properties such as linearity, additivity, and preservation of commutativity in the module structures.

Furthermore, to ensure a rigorous formulation, we apply known results on commuting mappings in rings, particularly those by Brešar (2004) and Khan (2019), to derive comparable outcomes for modules, the properties of derivations in rings and modules, with particular reference to works by Güven (2008) and Fitriani et al. (2025), and techniques from polynomial module theory to construct explicit examples of commuting mappings in module structures.

In the last step, we investigate the properties of commuting mappings on $T(+)N$. That is, conditions under which a finite sum of commuting mappings remains a commuting mapping, structural constraints that guarantee the linear combination of commuting mappings retains commutativity and identifying cases where centralizing mappings induce new commuting mappings within module structures.

3. RESULTS AND DISCUSSION

The concept of idealization is defined as follows. Let T be a commutative ring with identity, and let N be a unitary T module. The idealization of N, denoted by $T(+)N$, is defined as the direct sum $T \oplus N$ with addition given by

$$(\xi, \mu) + (\tau, \eta) = (\xi + \tau, \mu + \eta) \quad (1)$$

and multiplication defined as

$$(\xi, \mu)(\tau, \eta) = (\xi\tau, \xi\eta + \tau\mu) \quad (2)$$

$T(+)N$ equipped with the operations defined in 1 and 2, forms a commutative ring. Furthermore, T can be naturally embedded into $T(+)N$ via the map i from T to $T(+)N$, where $i(\xi) = (\xi, 0)$, for all $\xi \in$ T (Anderson and Winders, 2009).

Now, we recall the definition of a commuting map on a ring.

Definition 3.1. Let T be a ring and $Y \subseteq T$. A map $\alpha : Y \to T$ is said to be a commuting (on Y) if

$$\alpha(\xi)\xi = \xi\alpha(\xi) \tag{3}$$

for all $\xi \in Y$ (Brešar, 2004)

For any $\xi, \tau \in T$, the difference $\xi\tau - \tau\xi$ is referred to as the commutator, denoted by $[\xi, \tau]$. According to Equation 3, a map α is called a commuting map if $[\alpha(\xi), \xi] = 0$ for every $\xi \in T$. In the following proposition, we will show that the sum and scalar multiplication of commuting maps are also commuting maps.

Proposition 3.2. Let T be a ring, and $Y \subseteq T$. If α and δ are commuting maps on Y, then $\alpha + \delta$ and $\alpha\delta$, where

$$(\alpha + \delta)(\xi) = \alpha(\xi) + \delta(\xi) \tag{4}$$

and

$$(\alpha\delta)(\xi) = \alpha(\xi)\delta(\xi) \tag{5}$$

for all $\xi \in T$, are commuting maps on Y.

Proof. Let $\xi \in Y$. Based on Equation 4 and 5, we obtain:

$$\begin{aligned}(\alpha + \delta)(\xi)\xi &= (\alpha(\xi) + \delta(\xi))\xi \\ &= \alpha(\xi) + \delta(\xi)\xi \\ &= \xi\alpha(\xi) + \delta(\xi) \\ &= \xi((\alpha + \delta)\xi)\end{aligned}$$

$$\begin{aligned}(\alpha\delta)(\xi)\xi &= (\alpha(\xi)\delta(\xi))\xi \\ &= \alpha(\xi)\delta(\xi)\xi \\ &= \xi\alpha(\xi)\delta(\xi) \\ &= \xi(\alpha\delta)(\xi)\end{aligned}$$

This shows that $\alpha + \delta$ and $\alpha\ \delta$ are commuting maps on Y.

Now, we give $T(+)N$ as a T-module, where T is an arbitrary ring as follows.

Proposition 3.3. Let T be a ring with unity and N be an unitary T-module. Then $T(+)N$ equipped with coordinate-wise addition forms an Abelian group.

Proof. We will prove that $\langle T(+)N\rangle$ is an Abelian group.

1. Given any $(\xi, \mu), (\tau, \eta) \in$ T (+)N. We have $(\xi, \mu)+(\tau, \eta) = (\xi + \tau, \mu + \eta) \in T(+)N$.

2. There exist $(0_T, 0_N)$ so that $(0_T, 0_N) + (\xi, \mu) = (\xi, \mu) + (0_T, 0_N) = (\xi, \mu)$, for every $(\xi, \mu) \in T(+)N$.

3. Given any $(\xi, \mu) \in T(+)N$. There exists $(-\xi, -\mu) \in T \oplus N$ so that $(-\xi, -\mu) + (\xi, \mu) = (\xi, \mu) + (-\xi, -\mu) = (0_T, 0_N)$.

So, <T (+)N, +> is an Abelian group.

Proposition 3.4. Let T be a ring with unity and N be an unitary T-module. Then the direct sum $T(+)N$ equipped with coordinate-wise addition and scalar multiplication defined by

$$\kappa(\xi, \mu) = (\kappa\xi, \kappa\mu), \tag{6}$$

for every $\kappa \in$ T, $(\xi, \mu) \in T(+)N$, forms a T-module.

Proof. In the Proposition 3.3, we proved that $<T(+)N, +>$ forms an Abelian group. Now, we show that $T(+)N$ is a T-module.

1. Given any $\kappa \in$ T,; $(\xi, \mu), (\tau, \eta) \in$ T(+)N. Based on Equation 6, we obtain:

$$\begin{aligned}\kappa((\xi, \mu) + (\tau, \eta)) &= \kappa(\xi + \tau, \mu + \eta) \\ &= (\kappa(\xi + \tau), \kappa(\mu + \eta)) \\ &= (\kappa\xi + \kappa\tau, \kappa\mu + \kappa\eta) \\ &= (\kappa\xi, \kappa\mu) + (\kappa\tau, \kappa\eta) \\ &= \kappa(\xi, \mu) + \kappa(\tau, \eta)\end{aligned}$$

2. Given any $\kappa_1, \kappa_2 \in T$, ; $(\xi, \mu) \in T(+)N$:

$$\begin{aligned}(\kappa_1 + \kappa_2)(\xi, \mu) &= ((\kappa_1 + \kappa_2)\xi, (\kappa_1 + \kappa_2)\mu) \\ &= (\kappa_1\xi + \kappa_2\xi, \kappa_1\mu + \kappa_2\mu) \\ &= (\kappa_1\xi, \kappa_1\mu) + (\kappa_2\xi, \kappa_2\mu) \\ &= \kappa_1(\xi, \mu) + \kappa_2(\xi, \mu)\end{aligned}$$

3. Given any $\kappa_1, \kappa_2 \in T$, ; $(\xi, \mu) \in T(+)N$:

$$\begin{aligned}(\kappa_1\kappa_2)(\xi, \mu) &= ((\kappa_1\kappa_2)\xi, (\kappa_1\kappa_2)\mu) \\ &= (\kappa_1(\kappa_2\xi), \kappa_1(\kappa_2\mu)) \\ &= \kappa_1(\kappa_2\xi, \kappa_2\mu) \\ &= \kappa_1(\kappa_2(\xi, \mu))\end{aligned}$$

4. Given any $(\xi, \mu) \in$ T(+)N:

$$\begin{aligned}1(\xi, \mu) &= (1\xi, 1\mu) \\ &= (\xi, \mu)\end{aligned}$$

So, $T(+)N$ forms a T-module.

Now, recall the following definition of polynomial modules over polynomial ring $T[x]$. Let T be a ring and N be a module over T. We define $T[x]$ and $N[x]$ in 7 and 8.

$$T[x] = \{q : \mathbb{N} \cup \{0\} \to T \mid \text{supp}(q) \text{ finite}\} \tag{7}$$

and

$$N[x] = \{\sigma : \mathbb{N} \cup \{0\} \to N \mid \text{supp}(\sigma) \text{ finite}\} \tag{8}$$

Next, we define the following two operations:

$$(\sigma_1 + \sigma_2)(n) = \sigma_1(n) + \sigma_2(n), \tag{9}$$

and

$$(q\sigma)(n) = \sum_{k+l=n} q(k)\sigma(l), \tag{10}$$

for all $\sigma_1, \sigma_2 \in N[x]$, $q \in T[x]$, and $n \in \mathbb{Z}^+$. $N[x]$ is an $T[x]$ module with operations in 9 and 10 (Varadarajan, 2001). On the other hand, the structure of polynomial module $N[x]$ over a polynomial ring $T[x]$ can be expressed as Equation 11.

$$N[x] = \left\{ \sum_{l=0}^{k} n_l x^l \,\middle|\, k \in \mathbb{Z}^+,\ n_l \in N \right\}. \tag{11}$$

Then, according to Faisol et al. (2019), for any $\sum_{c=0}^{d} t_c x^c \in T[x]$ and $\sum_{l=0}^{k} n_l x^l \in N[x]$, the scalar multiplication is defined by

$$\left(\sum_{l=0}^{k} n_l x^l\right)\left(\sum_{c=0}^{d} t_c x^c\right) = \left(\sum_{v=0}^{k+d} s_v x^v\right), \tag{12}$$

where $s_v = \sum_{l+c=v} n_l t_c$. With operation in 12, $N[x]$ is a polynomial module over $T[x]$.

Before constructing the module $T(+)N$ as a module over T, we give the proof that $T(+)N$, with T an arbitrary ring and N a module over T, is a non-associative ring which is in the following proposition.

Proposition 3.5. Let T be a ring with 1 and N be an unitary T-module. Then $T(+)N$ with operations as follows:

$$(\xi, \mu) + (\tau, \eta) = (\xi + \tau,\ \mu + \eta) \tag{13}$$

$$(\xi, \mu)(\tau, \eta) = (\xi\tau,\ \xi\eta + \mu\tau) \tag{14}$$

Proof. We prove that $T(+)N$ is a non-associative ring with identity. In the Proposition 3.3, we proved that $\langle T(+)N, +\rangle$ forms an Abelian group. Now, given any (ξ, μ), (τ, η), (σ, ω) $\in T(+)N$. Based on Equation 13 and 14, we have:

$$\begin{aligned}(\xi, \mu)((\tau, \eta) + (\sigma, \omega)) &= (\xi, \mu)(\tau + \sigma,\ \eta + \omega)\\ &= (\xi(\tau + \sigma),\ \xi(\eta + \omega) + (\tau + \sigma)\mu)\\ &= (\xi\tau + \xi\sigma,\ \xi\eta + \xi\omega + \tau\mu + \sigma\mu)\\ &= (\xi\tau,\ \xi\eta + \tau\mu) + (\xi\sigma,\ \xi\omega + \sigma\mu)\\ &= (\xi, \mu)(\tau, \eta) + (\xi, \mu)(\sigma, \omega)\end{aligned}$$

$$\begin{aligned}[(\tau, \eta) + (\sigma, \omega)](\xi, \mu) &= (\tau + \sigma,\ \eta + \omega)(\xi, \mu)\\ &= ((\tau + \sigma)\xi,\ (\tau + \sigma)\mu + \xi(\eta + \omega))\\ &= (\tau\xi + \sigma\xi,\ \tau\mu + \sigma\mu + \xi\eta + \xi\omega)\\ &= (\tau\xi,\ \tau\mu + \xi\eta) + (\sigma\xi,\ \sigma\mu + \xi\omega)\\ &= (\tau, \eta)(\xi, \mu) + (\sigma, \omega)(\xi, \mu)\end{aligned}$$

We can conclude that $T(+)N$ is in a non-associative ring.

The multiplication operation on $T(+)N$ in Equation 14 is not associative. We can see this in the following explanation. Given any (ξ, μ), (τ, η), $(\sigma, \omega) \in T(+)N$. Based on Equation 14, we have:

$$\begin{aligned}(\xi, \mu)((\tau, \eta)(\sigma, \omega)) &= (\xi, \mu)((\tau\sigma, \tau\omega + \sigma\eta))\\ &= (\xi\tau\sigma, \xi(\tau\omega + \sigma\eta) + \tau\sigma\mu)\\ &= (\xi\tau\sigma, \xi\tau\omega + \xi\sigma\eta + \tau\sigma\mu).\end{aligned}$$

On the other hand,

$$\begin{aligned}((\xi, \mu)(\tau, \eta))(\sigma, \omega) &= (\xi\tau, \xi\eta + \tau\mu)(\sigma, \omega)\\ &= (\xi\tau\sigma, \xi\tau\omega + \sigma(\xi\eta + \tau\mu))\\ &= (\xi\tau\sigma, \xi\tau\omega + \sigma\xi\eta + \sigma\tau\mu).\end{aligned}$$

Note that $(\xi, \mu)((\tau, \eta)(\sigma, \omega))$ and $((\xi, \mu)(\tau, \eta))-(\sigma, \omega)$ not need to be equal, unless T is commutative. The next example illustrates this condition when T is not commutative.

Example 3.6

Consider $M_{2\times 2}(\mathbb{Z})$ as a module over itself. Choose $(\xi, \mu) = \left(\begin{bmatrix} 1 & 3 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 2 \\ -1 & 0 \end{bmatrix}\right)$, $(\tau, \eta) = \left(\begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}\right)$, $(\sigma, \omega) = \left(\begin{bmatrix} 2 & -2 \\ 1 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}\right) \in T(+)N$.

We have:

$$\begin{aligned}
&((\xi,\mu)(\tau,\eta))(\sigma,\omega)\\
&=\left(\left(\begin{bmatrix}1&3\\-1&0\end{bmatrix},\begin{bmatrix}-1&2\\-1&0\end{bmatrix}\right)\left(\begin{bmatrix}2&0\\1&3\end{bmatrix},\begin{bmatrix}2&1\\3&0\end{bmatrix}\right)\right)\\
&\quad\left(\begin{bmatrix}2&-2\\1&-1\end{bmatrix},\begin{bmatrix}2&1\\0&3\end{bmatrix}\right)\\
&=\left(\begin{bmatrix}1&3\\-1&0\end{bmatrix}\begin{bmatrix}2&0\\1&3\end{bmatrix},\begin{bmatrix}1&3\\-1&0\end{bmatrix}\begin{bmatrix}2&1\\3&0\end{bmatrix}\right.\\
&\quad\left.+\begin{bmatrix}2&0\\1&3\end{bmatrix}\begin{bmatrix}-1&2\\-1&0\end{bmatrix}\right)\left(\begin{bmatrix}2&-2\\1&-1\end{bmatrix},\begin{bmatrix}2&1\\0&3\end{bmatrix}\right)\\
&=\left(\begin{bmatrix}5&9\\-2&0\end{bmatrix},\begin{bmatrix}9&5\\-6&1\end{bmatrix}\right)\left(\begin{bmatrix}2&-2\\1&-1\end{bmatrix},\begin{bmatrix}2&1\\0&3\end{bmatrix}\right)\\
&=\left(\begin{bmatrix}5&9\\-2&0\end{bmatrix}\begin{bmatrix}2&-2\\1&-1\end{bmatrix},\begin{bmatrix}5&9\\-2&0\end{bmatrix}\begin{bmatrix}2&1\\0&3\end{bmatrix}\right.\\
&\quad\left.+\begin{bmatrix}2&-2\\1&-1\end{bmatrix}\begin{bmatrix}9&5\\-6&1\end{bmatrix}\right)\\
&=\left(\begin{bmatrix}19&-19\\-4&4\end{bmatrix},\begin{bmatrix}40&40\\11&2\end{bmatrix}\right).
\end{aligned}$$

On other hand

$$\begin{aligned}
(\xi,\mu)((\tau,\eta)(\sigma,\omega))&=(\begin{bmatrix}1&3\\-1&0\end{bmatrix},\begin{bmatrix}-1&2\\-1&0\end{bmatrix})\\
&\quad(\begin{bmatrix}2&0\\1&3\end{bmatrix},\begin{bmatrix}2&1\\3&0\end{bmatrix})\\
&\quad(\begin{bmatrix}2&-2\\1&-1\end{bmatrix},\begin{bmatrix}2&1\\0&3\end{bmatrix})\\
&=(\begin{bmatrix}1&3\\-1&0\end{bmatrix},\begin{bmatrix}-1&2\\-1&0\end{bmatrix})\\
&\quad(\begin{bmatrix}4&-4\\5&-5\end{bmatrix},\begin{bmatrix}5&4\\1&11\end{bmatrix})\\
&=(\begin{bmatrix}19&-19\\-4&4\end{bmatrix},\begin{bmatrix}8&45\\-5&6\end{bmatrix})
\end{aligned}$$

This shows that

$$(\xi,\mu)((\tau,\eta)(\sigma,\omega))\neq((\xi,\mu)(\tau,\eta))(\sigma,\omega)$$

and hence the multiplication operation on $T(+)N$ is not associative.

If T is commutative, we have the following property of $T(+)N$.

Corollary 3.7 Let T be a commutative ring with identity and N be a unitary T-module. Then $T(+)N$ with operations as follows:

$$(\xi,\mu)+(\tau,\eta)=(\xi+\tau,\mu+\eta) \tag{15}$$

$$(\xi,\mu)(\tau,\eta)=(\xi\tau,\xi\eta+\tau\mu), \tag{16}$$

for every $(\xi,\mu)+(\tau,\eta)\in T(+)N$, is a ring with identity.

Proof. We proved that $T(+)N$ is a non-associative ring with identity in the Proposition 3.5. Given any (ξ,μ), (τ,η), $(\sigma,\omega)\in T(+)N$. Based on Equation 15 and 16, we have:

$$\begin{aligned}
(\xi,\mu)((\tau,\eta)(\sigma,\omega))&=(\xi,\mu)((\tau\sigma,\tau\omega+\sigma\eta))\\
&=(\xi\tau\sigma,\xi(\tau\omega+\sigma\eta)+\tau\sigma\mu)\\
&=(\xi\tau\sigma,\xi\tau\omega+\xi\sigma\eta+\tau\sigma\mu)\\
&=(\xi\tau\sigma,\xi\tau\omega+\sigma\xi\eta+\sigma\tau\mu)\\
&=((\xi,\mu)(\tau,\eta))(\sigma,\omega).
\end{aligned}$$

This result shows that the multiplication operation on $T(+)N$ is an associative when T is commutative.

Furthermore, we give ideas for commuting and centralizing maps on modules. Furthermore, we provide some properties of this map on $T(+)N$. We begin discussions by defining centralizing and commuting maps on $T(+)N$ as a T-module. This definition is a generalization of Definition 3.1.

Definition 3.8 Let T be a ring, N be a T-module, and let $\alpha: T(+)N\to T(+)N$ be a map. We call α is a centralizing map on $T(+)N$ into $T(+)N$ if:

$$[\alpha(\xi,\mu),(\xi,\mu)]\in Z(T(+)N), \tag{17}$$

for all $(\xi,\mu)\in T(+)N$.

As a special case of a centralizing map, if $[\alpha(\xi,\mu),(\xi,\mu)]=0$ in Equation 17, we have the definition of a commuting map as follows.

Definition 3.9 Let T be a ring, N be a T-module, and $\alpha: T(+)N\to T(+)N$ be a map. We call α a commuting map of $T(+)N$ into $T(+)N$ if:

$$[\alpha(\xi,\mu),(\xi,\mu)]=0, \tag{18}$$

for all $(\xi,\mu)\in T(+)N$.

Based on Definition 3.9, every commuting map is a centralizing map on $T(+)N$ as a T-module. Moreover, α is a commuting map of $T(+)N$ into $T(+)N$ if and only if

$$\alpha(\xi,\mu)(\xi,\mu)-(\xi,\mu)\alpha(\xi,\mu)=0.$$

Let $\alpha(\xi,\mu)=(\tau,\eta)$, where $(\tau,\eta)\in T(+)N$. Based on Equation 18, we have:

$$\begin{aligned}
(\tau,\eta)(\xi,\mu)-(\xi,\mu)(\tau,\eta)&=0\\
(\tau\xi,\tau\mu+\xi\eta)-(\xi\tau,\xi\eta+\mu\tau)&=0\\
(\tau\xi-\xi\tau,\tau\mu+\xi\eta-\xi\eta-\mu\tau)&=0\\
(\tau\xi,\tau\mu+\xi\eta)&=(\xi\tau,\xi\eta+\mu\tau).
\end{aligned}$$

Next, we discuss the examples of commuting maps on $T(+)N$ in the following proposition.

Proposition 3.10 Let T be a ring and N be a module over T. The following mappings are commuting (centralizing) mapping on $T(+)N$.

1. $\alpha(\xi,\mu) = (0,0)$;
2. $\mathrm{id}(\xi,\mu) = (\xi,\mu)$;
3. $\delta(\xi,\mu) = (0,\mu)$;
4. $\gamma(\xi,\mu) = (\lambda,\mu), \quad \lambda \in T,$ for every $(\xi,\mu) \in T(+)N$.

Furthermore, γ is a commuting (centralizing) map when T is commutative.

Proof. It is easy to show that α, id, and δ are commuting maps. Now, we will prove that γ is also a commuting map when T is commutative. Let $(\xi,\mu) \in T(+)N$. Then, we have

$$\begin{aligned}[\gamma(\xi,\mu),(\xi,\mu)] &= (\lambda,\mu)(\xi,\mu) - (\xi,\mu)(\lambda,\mu)\\ &= (\lambda\xi, \lambda\mu + \xi\mu) - (\xi\lambda, \xi\mu + \lambda\mu)\\ &= (\lambda\xi - \xi\lambda, \lambda\mu + \xi\mu - \xi\mu - \lambda\mu)\\ &= (0,0), \quad \text{if } T \text{ is commutative.}\end{aligned}$$

Now, we construct $T[x](+)N[x]$ as follows:

$$T[x](+)N[x] = T[x] \oplus N[x] = \{(q,\sigma) \mid q \in T[x],\ \sigma \in N[x]\}.$$

Then, we define:

$$(q,\sigma) + (q',\sigma') = (q+q', \sigma+\sigma'), \tag{19}$$

where

$$(q+q')(n) = q(n) + q'(n), \quad \text{and} \quad (\sigma+\sigma')(n) = \sigma(n) + \sigma'(n),$$

for every $q, q' \in T[x]$, $\sigma, \sigma' \in N[x]$, and $n \in \mathbb{Z}^+$. With operation in 19, $T[x](+)N[x]$ form an Abelian group and hence $T[x](+)N[x]$ become a module over $T[x]$ (Varadarajan, 2001).

Furthermore, in the following example, we will define a commuting map using $T[x](+)N[x]$.

Example 3.11 Let T be the set of all invertible 2×2 matrices under matrix addition and matrix multiplication. Then

$$T[x] = \left\{ \sum_{i=1}^{n} A_i x^i \mid A_i \in T, n \in \mathbb{N} \right\}.$$

We define $\theta : T[x] \to T[x]$,

$$\theta\left(\sum_{i=0}^{n} a_i x^i\right) = S, \tag{20}$$

where

$$S = \begin{bmatrix} n & 0 \\ 0 & n \end{bmatrix} \in T,\ n \in \mathbb{Z},\ n \geq 1.$$

Based on Equation 20, it is easy to prove that

$$\sum_{i=0}^{n} a_i x^i \theta\left(\sum_{i=0}^{n} a_i x^i\right) = \theta\left(\sum_{i=0}^{n} a_i x^i\right) \sum_{i=0}^{n} a_i x^i, \tag{21}$$

for every $\sum_{i=0}^{n} a_i x^i \in T[x]$. Hence, θ is a commuting map on $T[x]$. Then, we define

$$\theta' : T[x](+)N[x] \to T[x](+)N[x], \text{ where}$$

$$\theta'\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right) = \left(\theta\left(\sum_{i=0}^{n} a_i x^i\right), \sum_{j=0}^{k} m_j x^j\right).$$

for every $\sum_{i=0}^{n} a_i x^i \in T[x]$ and $\sum_{j=0}^{k} m_j x^j \in N[x]$, we will show that θ' is a commuting map on $T(+)N$. Given any $\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right) \in T[x](+)N[x]$. Based on Equation 21, we have:

$$\begin{aligned}&\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\theta'\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\\ &= \left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\left(\theta\left(\sum_{i=0}^{n} a_i x^i\right), \sum_{j=0}^{k} m_j x^j\right)\\ &= \left(\sum_{i=0}^{n} a_i x^i\, \theta\left(\sum_{i=0}^{n} a_i x^i\right), \sum_{i=0}^{n} a_i x^i \sum_{j=0}^{k} m_j x^j\right.\\ &\quad \left. + S\left(\sum_{j=0}^{k} m_j x^j\right)\right)\end{aligned}$$

On the other hand,

$$\begin{aligned}&\theta'\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\\ &= \left(S, \sum_{j=0}^{k} m_j x^j\right)\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\\ &= \left(\sum_{i=0}^{n} a_i x^i S, \left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right) + S\left(\sum_{j=0}^{k} m_j x^j\right)\right)\end{aligned}$$

So, we have

$$\begin{aligned}&\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\theta'\left(\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\right)\\ &= \theta'\left(\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\right)\left(\sum_{i=0}^{n} a_i x^i, \sum_{j=0}^{k} m_j x^j\right)\end{aligned}$$

This result shows that θ' is a commuting map on $T[x](+)N[x]$.

Based on Example 3.11, the following theorem shows the connection between the commuting maps on ring T and the commuting maps on $T(+)N$ as a T-module.

Theorem 3.12. Let T be a ring and N be an T-module. If α is a commuting map on T, then

$$\alpha' : T(+)N \to T(+)N, \text{ where } \alpha'(\xi,\mu) = (\alpha(\xi),\mu) \tag{22}$$

is a commuting map on $T(+)N$.

Proof. Assume that α is a commuting map on ring T. Let $(\xi, \mu) \in T(+)N$. Based on Equation 22, we have:

$$\begin{aligned}[\alpha'(\xi, \mu), (\xi, \mu)] &= [(\alpha(\xi), \alpha(\mu)), (\xi, \mu)] \\ &= (\alpha(\xi), \mu)(\xi, \mu) - (\xi, \mu)(\alpha(\xi), \mu) \\ &= (\alpha(\xi)\xi, \alpha(\xi)\mu + \xi\mu) - (\xi\alpha(\xi), \\ &\quad \xi\mu + \alpha(\xi)\mu) \\ &= (\alpha(\xi)\xi - \xi\alpha(\xi), \alpha(\xi)\mu + \xi\mu - \\ &\quad \xi\mu - \alpha(\xi)\mu) \\ &= (0, 0),\end{aligned}$$

since α is a commuting map on T. Hence, α' is commuting map on $T(+)N$.

We will give the properties of commuting maps on $T(+)N$ in the following propositions. In the Proposition 3.13, we investigate the properties of a commuting map in ring $(T(+)N)^n = \{(\xi_1, \mu_1), (\xi_2, \mu_2), \ldots, (\xi_n, \mu_n) \mid \xi_i \in T, \mu_i \in N$ for every $i = 1, 2, \ldots, n, n \in \mathbb{N}\}$.

Proposition 3.13. Let T be a ring. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are commuting maps on $T(+)N$, then $\oplus\alpha_i$ is a commuting map on $(T(+)N)^n$.

Assume that $\alpha_1, \alpha_2, \ldots, \alpha_n$ are commuting maps on $T(+)N$. We define $\oplus_{i=1}^n\alpha_i$ from $(T(+)N)^n$ to itself, where

$$\oplus_{i=1}^n\alpha_i((\xi_1, \mu_1), \ldots, (\xi_n, \mu_n)) = (\alpha_1(\xi_1, \mu_1), \ldots, \alpha_n(\xi_n, \mu_n)). \tag{23}$$

We will show that $\oplus_{i=1}^n\alpha_i$ is a commuting map on $(T(+)N)^n$.

Let $((\xi_1, \mu_1), (\xi_2, \mu_2), \ldots, (\xi_n, \mu_n)) \in (T(+)N)^n$. Based on Equation 23, we have:

$$\begin{aligned}&\oplus_{i=1}^n\alpha_i((\xi_1, \mu_1), \ldots, (\xi_n, \mu_n)) \\ &\quad((\xi_1, \mu_1), \ldots, (\xi_n, \mu_n)) \\ &= \oplus_{i=1}^n(\alpha_i((\xi_1, \mu_1), \ldots, (\xi_n, \mu_n)) \\ &\quad((\xi_1, \mu_1), \ldots, (\xi_n, \mu_n))) \\ &= ((\xi_1, \mu_1), (\xi_2, \mu_2), \ldots, (\xi_n, \mu_n)) \oplus_{i=1}^n \alpha_i((\xi_1, \mu_1), \\ &\quad(\xi_2, \mu_2), \ldots, (\xi_n, \mu_n)).\end{aligned}$$

This shows that $\oplus\alpha_i$ is a commuting map on $(T(+)N)^n$.

In the following proposition, we show that we can construct a new commuting map in a ring T, a finite sum of commutation maps that is also a commuting map in a ring T.

Proposition 3.14. Let T be a ring. If $\alpha_1, \alpha_2, \ldots, \alpha_n$ are commuting maps on $T(+)N$, then $\sum_{i=1}^n \alpha_i$, where

$$\left(\sum_{i=1}^n \alpha_i\right)(\xi, \mu) = \sum_{i=1}^n(\alpha_i(\xi, \mu)) \tag{24}$$

is a commuting map on $T(+)N$.

Proof. Given any $(\xi, \mu) \in T(+)N$. Based on Equation 24, we have:

$$\begin{aligned}(\xi, \mu)\left(\sum_{i=1}^n \alpha_i\right)(\xi, \mu) &= (\xi, \mu)\left(\sum_{i=1}^n(\alpha_i(\xi, \mu))\right) \\ &= (\xi, \mu)(\alpha_1(\xi, \mu)) + \ldots + \\ &\quad(\xi, \mu)(\alpha_n(\xi, \mu)) \\ &= (\alpha_1(\xi, \mu))(\xi, \mu) + \ldots + \\ &\quad(\alpha_n(\xi, \mu))(\xi, \mu) \\ &= (\alpha_1 + \ldots + \alpha_n)(\xi, \mu)(\xi, \mu) \\ &= \left(\sum_{i=1}^n \alpha_i\right)(\xi, \mu)(\xi, \mu)\end{aligned}$$

This shows that $\sum_{i=1}^n \alpha_i$ is a commuting map on $T(+)N$.

Now, in Proposition 3.15 we prove that the linear combination of commuting maps in a ring T is also a commuting map under certain conditions.

Proposition 3.15. Let $n \geq 1$ be a fixed integer and T be a ring. If α_i is a commuting map on $T(+)N$, $i = 1, 2, \ldots, n$ and

$$\kappa(\xi, \mu) = (\xi, \mu)\kappa, \tag{25}$$

for every $\kappa \in T$, then $\sum_{i=1}^n \kappa_i\alpha_i$ is a commuting map on $T(+)N$.

Proof. Given any $\kappa_i \in T$ for $i = 1, 2, \ldots, n$ and $(\xi, \mu) \in T(+)N$. According to Equation 25, we have:

$$\begin{aligned}(\xi, \mu)\left(\sum_{i=1}^n \kappa_i\alpha_i\right)(\xi, \mu) &= (\xi, \mu)(\kappa_1\alpha_1(\xi, \mu) + \cdots + \\ &\quad\kappa_n\alpha_n(\xi, \mu)) \\ &= \kappa_1(\xi, \mu)\alpha_1(\xi, \mu) + \cdots + \\ &\quad\kappa_n(\xi, \mu)\alpha_n(\xi, \mu) \\ &= \kappa_1\alpha_1(\xi, \mu)(\xi, \mu) + \cdots + \\ &\quad\kappa_n\alpha_n(\xi, \mu)(\xi, \mu) \\ &= \left(\sum_{i=1}^n \kappa_i\alpha_i\right)(\xi, \mu)(\xi, \mu).\end{aligned}$$

This shows that $\sum_{i=1}^n \kappa_i\alpha_i$ is a commuting map on $T(+)N$.

4. CONCLUSIONS

If T is an arbitrary ring, then $T(+)N$ is a non asscociative ring. By defining the scalar multiplication operation, we get $T(+)N$ is a module over R. We can define centralizing and commuting maps in $T(+)N$ by using the multiplication operation in idealization concept. This definition is a generalization of centralizing and commuting mapping in rings. Furthermore, if α is a commuting mapping on a ring T, then a corresponding commuting mapping α' can be defined on the module by utilizing α. Moreover, we proved that the finite sum of commuting mappings is also a commuting mapping and that a linear combination of commuting mappings is also a commuting mapping under certain conditions.

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