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Research Paper

Delta Degree-Based Indices of Prime Coprime Graph for Integers Modulo Group

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Abstract
Research on prime coprime graphs of finite groups has largely focused on structural properties, spectra, and classical topological indices, with limited attention given to delta degree-based indices. To address this gap, this study investigates delta degree-based topological indices of the prime coprime graph constructed on the group of integers modulo n, Z_n. In this graph, the vertices correspond to the elements of Z_n, and two distinct vertices are adjacent if and only if the greatest common divisor of their orders is either 1 or a prime number. In the present work, the focus lies on computing and analyzing several delta degree-based topological indices that are obtained by incorporating the concept of delta degree into classical topological indices, including the delta first Zagreb index, the delta second Zagreb index, the delta hyper Zagreb index, and the delta forgotten index. The methodology involves deriving formulas for these delta-based indices for various values of n, supported by systematic computations and data tabulation. Beyond purely algebraic computation, statistical tools are employed to investigate the relationships between different indices. In particular, a comparative distribution analysis is conducted to determine whether pairs of indices exhibit similar patterns of variability using the Levene test.

Keywords
Prime Coprime Graph, Delta Degree, Topological Indices, Integers Modulo Group

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1. INTRODUCTION

The topological indices are numerical values derived from mathematical computations related to the graph structure. Numerous topological indices have been delineated in theoretical chemistry and employed in diverse applications, particularly in QSPR/QSAR research (Gutman and Polansky, 2012; Kulli, 2018). A variety of topological indices have provided detailed information regarding the structural characteristics of benzenoid graphs. This information can be found in Asghar (2025). The chemical composition of pentacene (Ahmad et al., 2025), dendrimers (Gao et al., 2018), graphene (Kwun et al., 2019), vanadium carbide (Wei et al., 2023), metalorganic framework (Rosary, 2023), fullerene cage network (Ahmad et al., 2023), and oxide and honeycomb networks (Kulli et al., 2023) are examined through several topological indices.

Numerous forms of topological indices exist, such as those based on vertex degree, edge distance, or a combination of both vertex degree and edge distance. Researchers have made numerous modifications to topological indices based on vertex degree, such as the reverse index (Ediz and Cancan, 2016) and the revan index (Kulli et al., 2023). Research concerning the reverse index of various topological indices has been extensively conducted in both mathematics (Gowtham and Husin, 2023a) and chemistry (Gowtham and Husin, 2023b). The alteration involves adjusting the degrees by utilizing the maximum and/or minimum degrees and adding or removing 1. The delta index is another modified topological index. The delta index was initially presented by Kulli (2021a). The vertices in the index formula are adjusted by deducting the minimum node degree and incrementing by one. Presently, investigations concerning the delta index of various topological indices, specifically Banhatti and Sombor, remain mostly focused on applications in chemistry, including nanotubes (Kulli, 2021b), as well as silicate and hexagonal networks (Kulli, 2021a). In summary, there is limited research concerning the delta index within the discipline of mathematics.

In mathematics, particularly in algebraic graphs, the computation of topological indices is intrinsically linked to graphs. Graphs frequently analyzed are those whose vertices originate from algebraic structures, including groups and rings. More terminologies of a graph defined on a group can be found in (Romdhini et al., 2024, 2025) and the algebraic discussion in (Fitriani et al., 2025). Saini et al. (2021) derived an explicit formula for the vertex degree in the co-prime order graph of finite Abelian and dihedral groups, and further examined its Laplacian spectrum. Saini et al. (2021) investigated structural properties and several topological indices of the co-prime order graph of finite Abelian p-groups, including the Wiener, Hyper-Wiener, first and second Zagreb, Schultz, Gutman, and eccentric connectivity indices. The study has been extended from the Laplacian matrix spectrum to other degree-based matrices (Saini et al., 2025). Some conditions have been investigated under which the co-prime order graphs of finite groups, particularly Abelian groups and permutation groups, can be characterized as divisor graphs (Saini et al., 2024).

This present article examines the prime coprime graph when the vertices consist of all elements of a group. Two distinct vertices of a group G are considered adjacent if and only if the greatest common divisor of their orders is either prime or equal to 1 (Adhikari and Banerjee, 2021). Several research studies concerning this graph and its topological index pertain to reverse topological indices (Abdurahim et al., 2025a), and its spectral properties (Romdhini and Nawawi, 2025). The relationship between the Zagreb index and prime coprime graphs within the integer group of prime powers, namely $n = p^k$, has also been examined (Abdurahim et al., 2025b). With limited attention given to delta degree-based indices, therefore, this article examines the relationship between delta degree-based indices and prime coprime graphs of the integer group, drawing on concepts from Kulli (2021b) and Abdurahim et al. (2025b). More specifically, this article presents the delta indices of the first and second Zagreb, hyper Zagreb, and forgotten indices for prime coprime graphs on the group of integers modulo $2p^k$, where p is a prime and $k \geq 2$ is an integer. Eventually, we show a comparison analysis using the Levene test to determine whether the two indices exhibit similar value distributions.

This paper is organized as follows. In Section 2, we review the necessary theoretical background and preliminaries on prime coprime graphs and delta degree-based indices. In Section 3, we establish the main results by deriving explicit formulas for the delta degree-based topological indices of the prime coprime graph on $\mathbb{Z}_n$. We also apply statistical methods, including a comparative distribution analysis using the Levene test, to examine the relationships among different indices. Finally, Section 4 concludes the paper with a summary of findings and possible directions for future research.

2. EXPERIMENTAL SECTION

We now present the fundamental definitions and notation used in this research. We will begin by presenting the notation, along with its definition, in the Table 1 that follows.

Table 1. Notation and its Definition

Symbol	Definition
G	graph
$V(G)$	set of vertices of G
$E(G)$	set of edges of G
$\deg(u)$	degree of vertex u
δ_u	delta degree of vertex u
δ_G	minimum degree of vertices in G
$S(G)$	Sombor index of G
$M_1(G)$	first Zagreb index of G
$M_2(G)$	second Zagreb index of G
$HM(G)$	hyper Zagreb index of G
$\delta S(G)$	delta Sombor index of G
$\delta M_1(G)$	delta first Zagreb index of G
$\delta M_2(G)$	delta second Zagreb index of G
$\delta HM(G)$	delta hyper Zagreb index of G
$\delta F(G)$	delta forgotten index of G
$\mathbb{Z}_n$	group of integers modulo n
$o(u)$	order of u in $\mathbb{Z}_n$
$\Gamma_{\mathbb{Z}_n}$	prime coprime graph of $\mathbb{Z}_n$
p	prime number
k	integer greater than or equal to 2
$\Gamma(\mathbb{Z}_{2p^k})$	prime coprime graph for $\mathbb{Z}_n$, where $n = 2p^k$
$V(\Gamma(\mathbb{Z}_{2p^k}))$	set of vertices of $\Gamma(\mathbb{Z}_{2p^k})$
$E(\Gamma(\mathbb{Z}_{2p^k}))$	set of edges of $\Gamma(\mathbb{Z}_{2p^k})$

Let G be a graph. The definition of the delta Sombor index of graph G is motivated by the average Sombor Index and was introduced by Kulli (2021b). This definition was motivated by the classical Sombor index of a graph G, Equation 1, originally introduced by Gutman (2021) as

$$S(G) = \sum_{uv \in E(G)} \sqrt{(\deg(u))^2 + (\deg(v))^2}. \tag{1}$$

The first delta-based index introduced was the delta Sombor index (Equation 2 and Equation 3), it is defined for G as

$$\delta S(G) = \sum_{uv \in E(G)} \sqrt{(\deg(u) - \delta(G) + 1)^2 + (\deg(v) - \delta(G) + 1)^2}. \tag{2}$$

Let $\delta_u = \deg(u) - \delta(G) + 1$ and $\delta_v = \deg(v) - \delta(G) + 1$, then

$$\delta S(G) = \sum_{uv \in E(G)} \sqrt{(\delta_u)^2 + (\delta_v)^2}. \tag{3}$$

Similarly, we can establish a delta index for analogous topological indices. In particular, the classical Zagreb indices, comparising the first Zagreb (Gutman and Das, 2004) (Equation 4), second Zagreb (Das et al., 2015) (Equation 5), and hyper Zagreb(Shirdel et al., 2013) (Equation 6) indices, are defined as follows:

$$M_1(G) = \sum_{v \in V(G)} (\deg(v))^2 \quad (4)$$

$$M_2(G) = \sum_{uv \in E(G)} \deg(u) \cdot \deg(v) \quad (5)$$

$$HM(G) = \sum_{uv \in E(G)} (\deg(u) + \deg(v))^2 . \quad (6)$$

The corresponding delta versions of these indices, Equations 7, 8, 9, and 10, are formally introduced in the following definition.

Definition 2.1. The delta index for the first Zagreb, second Zagreb, hyper Zagreb, and forgotten indices for G are successively defined as follows:

$$\delta M_1(G) = \sum_{v \in V(G)} (\delta_v)^2 \quad (7)$$

$$\delta M_2(G) = \sum_{uv \in E(G)} \delta_u \cdot \delta_v \quad (8)$$

$$\delta HM(G) = \sum_{uv \in E(G)} (\delta_u + \delta_v)^2 \quad (9)$$

$$\delta F(G) = \sum_{v \in V(G)} (\delta_v)^3 . \quad (10)$$

The above indices in this study will be applied to the prime coprime graphs, which were defined by Adhikari and Banerjee (2021). The prime coprime graph of integers modulo group, $\mathbb{Z}_n$, is defined as the graph with two adjacent distinct vertices satisfy $\gcd(o(u), o(v)) = p$ or $\gcd(o(u), o(v)) = 1$. It should be noted that this paper focuses on the group $\mathbb{Z}_n$, where $n = 2p^k$, p is a prime, and $k \geq 2$ integer. Therefore, in further discussion, we denote $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ as the prime coprime graph for $\mathbb{Z}_n$, where $n = 2p^k$.

It has been established that the prime coprime graphs on the group of integers modulo is a simple connected graph without loops or multiple edges (Abdurahim et al., 2025a), which makes it suitable for the application of delta degree-based indices.

In this study, we use the group of integers modulo n with addition modulo n, denoted by $(\mathbb{Z}_n, +_n)$. For simplicity, throughout this paper $\mathbb{Z}_n$ refers to $(\mathbb{Z}_n, +_n)$. In the group $\mathbb{Z}_n$, the order of an element a denoted by $o(a)$ is defined as the smallest positive integer m such that $a^m = 0$. Equivalently, the order of a can be expressed as $o(a) = \dfrac{n}{\gcd(a, n)}$ (Dummit and Foote, 2004). On the other hand, since $lcm(a, n) \cdot \gcd(a, n) = a \cdot n$, we also have $o(a) = \dfrac{lcm(a, b)}{n}$ (Rosen, 2011). Therefore, in proving the order of vertices, we adopt the formula $o(a) = \dfrac{lcm(a, b)}{n}$, which simplifies the presentation of the proofs. We also consider the vertex set partition of $\mathbb{Z}_n$ where $n = 2p^k$, p an odd prime and $k \geq 2$ an integer, as follows:

$$\begin{aligned} V_1 &= \left\{2m \cdot p^{k-1} \mid 0 \leq m < p\right\} \cup \left\{p^k\right\}; \\ V_2 &= \left\{(2m-1)p^{k-1} \mid 1 \leq m \leq p\right\} \setminus \left\{p^k\right\}; \\ V_3 &= \left\{2m \mid 1 \leq m < p^k\right\} \setminus \left\{2n \cdot p^{k-1} \mid 1 \leq n < p\right\}; \\ V_4 &= \left\{2m-1 \mid 1 \leq m \leq p^k\right\} \setminus \left\{(2n-1)p^{k-1} \mid 1 \leq n \leq p\right\}, \end{aligned} \quad (11)$$

and $\mathbb{Z}_n = V_1 \cup V_2 \cup V_3 \cup V_4$.

Let us see an illustration for a group $\mathbb{Z}_{2 \cdot 3^2}$. We can determine the order of elements of $\mathbb{Z}_{2 \cdot 3^2}$ as $o(0) = o(6) = o(9) = o(12) = 17$, $o(3) = o(15) = 10$, $o(2) = o(4) = o(8) = o(10) = o(14) = o(16) = 6$, and $o(1) = o(5) = o(7) = o(11) = o(13) = o(17) = 4$. According to the definition of a prime coprime graph, we derive a prime coprime graph from the group $\mathbb{Z}_{2 \cdot 3^2}$ as illustrated in Figure 1.

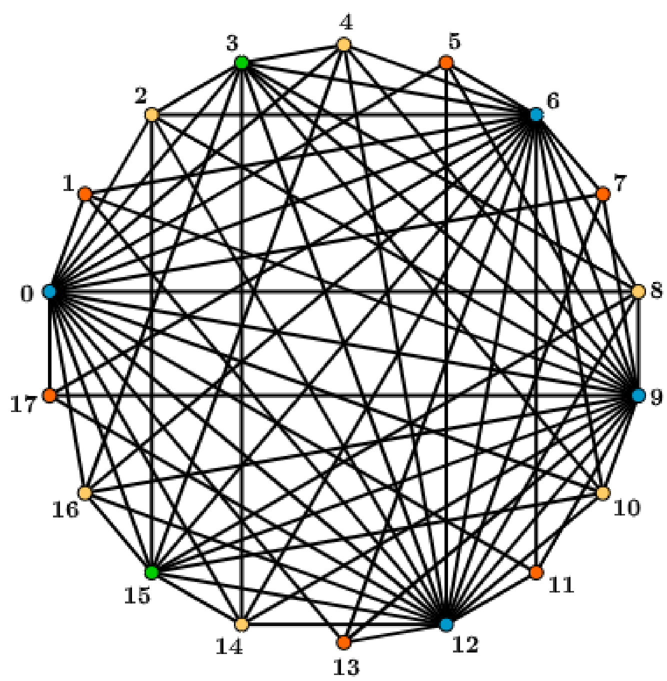

Figure 1. Prime Coprime Graph of Group $\mathbb{Z}_{18}$

3. RESULTS AND DISCUSSION

Since the delta index value depends on the degree of vertices, we need to determine it as stated in the following theorem.

Theorem 3.1 Given $\mathbb{Z}_n$ is a group with $n = 2p^k$, p is odd prime and $k \geq 2$ is an integer. For all $v \in V(\Gamma(\mathbb{Z}_n))$, then

$$o(v) = \begin{cases} 2p^k - 1 & , v \in V_1 \\ p^k + 1 & , v \in V_2 \\ 2p & , v \in V_3 \\ p + 1 & , v \in V_4 . \end{cases} \quad (12)$$

The first step in this proof is to determine the order of element v in V_1, V_2, V_3, and V_4 (Equation 11), where the order of v is $o(v) = \dfrac{lcm(v, 2p^k)}{v}$.

1. The first case, we construct the vertex set which is the element is an even multiple of p^{k-1}, and we can write as $\{2m \cdot p^{k-1} \mid m \in \mathbb{Z}, 1 \le m < p\}$. Taking an element v in this set, the order of v is
$$o(v) = \frac{lcm(2m \cdot p^{k-1}, 2p^k)}{2m \cdot p^{k-1}} = \frac{2m \cdot p^k}{2m \cdot p^{k-1}} = p. \quad (13)$$
Subsequently, in a similar manner, the order of p^k is $o(p^k) = 2$. Additionally, $o(0) = 1$. Consequently, the order of v_1 is either 1 or prime for each $v_1 \in V_1$ where $V_1 = \{2m \cdot p^{k-1} \mid m \in \mathbb{Z}, 0 \le m < p\} \cup \{p^k\}$.
2. In the second case, a set of vertices will be constructed that constitutes an odd multiple of p^{k-1}, specifically $\{(2m-1) \cdot p^{k-1} \mid m \in \mathbb{Z}, 1 \le m \le p\}$. This element v of this set has the order:
$$o(v) = \frac{lcm((2m-1) \cdot p^{k-1}, 2p^k)}{(2m-1) \cdot p^{k-1}} = \frac{2(2m-1)p^k}{(2m-1) \cdot p^{k-1}} = 2p. \quad (14)$$
Since $p^k \in \{(2m-1) \cdot p^{k-1} \mid m \in \mathbb{Z}, 1 \le m \le p\}$ and $o(p^k) = 2$, then $o(v_2) = 2p^k$ for every $v_2 \in V_2$, where $V_2 = \{(2m-1) \cdot p^{k-1} \mid m \in \mathbb{Z}, 1 \le m \le p\} \setminus \{p^k\}$.
3. In the third case, a set of vertices will be constructed from even numbers, namely $\{2m \mid m \in \mathbb{Z}, 1 \le m < p^k\}$, having the order
$$o(v) = \frac{lcm(2m, 2p^k)}{2m} = \frac{2mp^k}{2m} = p^k. \quad (15)$$
However, even numbers that are multiples of p^{k-1}, $\{2n \cdot p^{k-1} \mid n \in \mathbb{N}, 1 \le n < p\}$, have order $o(v) = p$. Thus, $o(v_1) = 2p^k$ for every $v_3 \in V_3$ where $V_3 = \{2m-1 \mid m \in \mathbb{Z}, 1 \le m \le p^k\}$ $\setminus \{(2n-1) \cdot p^{k-1} \mid n \in \mathbb{N}, 1 \le n \le p\}$.
4. In the fourth case, a collection of vertices will be established, comprising an odd number, specifically $\{2m-1 \mid m \in \mathbb{Z}, 1 \le m \le p^k\}$, which possesses the order
$$o(v) = \frac{lcm(2m-1, 2p^k)}{2m-1} = \frac{2(2m-1)p^k}{2m-1} = 2p^k. \quad (16)$$
For odd numbers that are multiples of p^{k-1}, the set $\{(2n-1) \cdot p^{k-1} \mid n \in \mathbb{N}, 1 \le n \le p\}$ has order $o(v) = 2p$. Hence, $o(v_4) = 2p^k$ for every $v_4 \in V_4$ where $V_4 = \{2m-1 \mid m \in \mathbb{Z}, 1 \le m \le p^k\}$ $\setminus \{(2n-1) \cdot p^{k-1} \mid n \in \mathbb{N}, 1 \le n \le p\}$.

The cardinality of each vertex set is as follows: $|V_1| = p+1$, $|V_2| = p-1$, $|V_3| = (p^k-1)-(p-1) = p^k - p$, and $|V_4| = p^k - p$. According to the concept of a prime coprime graph, every vertex in V_1 is connected to all vertices in the graph. Consequently, we get $\deg(v_1) = |V_1| + |V_2| + |V_3| + |V_4| - 1 = 2p^k - 1$. Moreover, each vertex in V_2 is connected to all vertices in V_1 and V_3, therefore $\deg(v_2) = |V_1| + |V_3| = p^k + 1$. Consequently, each vertex in V_3 is connected to vertices in V_1 and V_2, resulting in $\deg(v_3) = |V_1| + |V_2| = 2p$. Ultimately, every vertex in V_4 is next to a vertex in V_1. Consequently, we obtain $\deg(v_4) = |V_1| = p + 1$.

Let us consider the group $\mathbb{Z}_{2 \cdot 3^2}$. Based on Theorem 3.1, Equation 12, the vertex set is partitioned as follows: $V_1 = \{0, 6, 9, 12\}$, $V_2 = \{3, 15\}$, $V_3 = \{2, 4, 8, 10, 14, 16\}$, dan $V_4 = \{1, 5, 7, 11, 13, 17\}$. Therefore, the orders of the corresponding vertices are given by $o(0) = o(6) = o(9) = o(12) = 17$, $o(3) = o(15) = 10$, $o(2) = o(4) = o(8) = o(10) = o(14) = o(16) = 6$, and $o(1) = o(5) = o(7) = o(11) = o(13) = o(17) = 4$.

Theorem 3.1 Is specified for odd primes, since in the case $p = 2$ the vertex degree can be derived from Theorem 2.1 in (Abdurahim et al., 2025b). However, because $n = 2p^k$, when $p = 2$ this reduces to $n = 2^{k+1}$ as in Theorem 2.1. Therefore, the degree of a vertex in the prime coprime graph of the group $\mathbb{Z}_{2^{k+1}}$ is given by

$$\deg(v) = \begin{cases} 2^{k+1} - 1 & , v \in V_1 \\ 2 & , v \in V_2. \end{cases} \quad (17)$$

where $V_1 = \{0, 2^{k-1}\}$, and $V_2 = \{1, 2, \ldots, 2^{k-1} - 1, 2^{k-1} + 1, \ldots, 2^k - 1\}$.

Next, we need to determine the number of edges in the graph as a result required for the discussion of the next theorem.

Theorem 3.2 The number of edges in $\Gamma(\mathbb{Z}_{2p^k})$ is

$$\left|E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)\right| = \frac{1}{2}\left(6p^{k+1} + 2p^k - 3p^2 - p - 2\right). \quad (18)$$

Proof. The number of edges in a graph is equivalent to half the total of the degrees of all vertices (Gunderson and Rosen, 2010). Consequently, we derive

$$\begin{aligned}
\left|E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)\right| &= \frac{1}{2} \sum_{v \in V\left(\mathbb{Z}_{2p^k}\right)} \deg(v) \\
&= \frac{1}{2}\left(\sum_{v_1 \in V\left(\mathbb{Z}_{2p^k}\right)} \deg(v_1) + \sum_{v_1 \in V\left(\mathbb{Z}_{2p^k}\right)} \deg(v_2) + \right. \\
&\quad \left. \sum_{v_2 \in V\left(\mathbb{Z}_{2p^k}\right)} \deg(v_3) + \sum_{v_3 \in V\left(\mathbb{Z}_{2p^k}\right)} \deg(v_4) \right) \\
&= \frac{1}{2}\left((p+1) \cdot \left(2p^k - 1\right) + (p-1) \cdot \left(p^k + 1\right) + \left(p^k - p\right) \cdot 2p + \right. \\
&\quad \left. \left(p^k - p\right) \cdot (p+1)\right) \\
&= \frac{1}{2}\left(2p^{k+1} + 2p^k - p - 1 + p^{k+1} + p - p^k - 1 + 2p^{k+1} - 2p^2 + \right. \\
&\quad \left. p^{k+1} + p^k - p^2 - p\right)
\end{aligned}$$
$$\left|E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)\right| = \frac{1}{2}\left(6p^{k+1} + 2p^k - 3p^2 - p - 2\right).$$

It is now time to devise a formula for the delta-based indices. This formula is presented in Theorems 3.3, 3.4, 3.5.

Theorem 3.3 The delta first Zagreb index of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is

$$\delta M_1\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) = 5p^{2k+1} + 3p^{2k} - 5p^{k+2} - 4p^{k+1} - 5p^k + p^3 + 5p. \quad (19)$$

proof. This proof partitions the vertex set of the prime coprime graph $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ into four subsets, denoted as V_1, V_2, V_3, and V_4. By Theorem 3.1, Equation 12, we have $\delta_{v_1} = 2p^k - p - 1$, $\delta_{v_2} = p^k - p + 1$, $\delta_{v_3} = p$, and $\delta_{v_4} = 1$ for every $v_1 \in V_1, v_2 \in V_2$, $v_3 \in V_3$, and $v_4 \in V_4$.

$$\begin{aligned}
\delta M_1\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= \sum_{v \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} (\delta_v)^2 \\
&= \sum_{v_1 \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_1}\right)^2 + \sum_{v_2 \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_2}\right)^2 \\
&\quad + \sum_{v_3 \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_3}\right)^2 + \sum_{v_4 \in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_4}\right)^2 \\
&= (p+1)\left(2p^k - p - 1\right)^2 + (p-1)\left(p^k - p + 1\right)^2 \\
&\quad + \left(p^k - p\right)(p)^2 + \left(p^k - p\right)(1)^2 \\
&= (p+1)\left(4p^{2k} - 4p^{k+1} - 4p^k + p^2 + 2p + 1\right) + (p-1) \\
&\quad \left(p^{2k} - 2p^{k+1} + 2p^k + p^2 - 2p + 1\right)\left(p^{k+2} - p^3\right) + \left(p^k - p\right) \\
&\quad \left(p^{2k+1} - 2p^{k+2} + 2p^{k+1} + p^3 - 2p^2 + p - p^{2k} + 2p^{k+1} - 2p^k - p^2 + 2p - 1\right) \\
&\quad \left(p^{k+2} - p^3\right) + \left(p^k - p\right) \\
\delta M_1\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= 5p^{2k+1} + 3p^{2k} - 5p^{k+2} - 4p^{k+1} - 5p^k + p^3 + 5p
\end{aligned}$$

Thus, the general formula for the first delta Zagreb index on the prime coprime graph $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is

$$\delta M_1\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) = 5p^{2k+1} + 3p^{2k} - 5p^{k+2} - 4p^{k+1} - 5p^k + p^3 + 5p \quad (20)$$

The delta degree of each vertex in $\Gamma(\mathbb{Z}_{18})$ (Figure 1) is given by $\delta_0 = \delta_6 = \delta_9 = \delta_{12} = 14$, $\delta_3 = \delta_{15} = 7$, $\delta_2 = \delta_4 = \delta_8 = \delta_{10} = \delta_{14} = \delta_{16} = 3$, $\delta_1 = \delta_5 = \delta_7 = \delta_{11} = \delta_{13} = \delta_{17} = 1$. Using the definition of the delta first Zagreb index, the value derived from Figure 1 is obtained as follows.

$$\begin{aligned}
\delta M_1\left(\Gamma\left(\mathbb{Z}_{2\cdot 3^2}\right)\right) &= 4 \cdot (\delta_0)^2 + 2 \cdot (\delta_3)^2 + 6 \cdot (\delta_2)^2 + 6 \cdot (\delta_1)^2 \\
&= 4 \cdot 14^2 + 2 \cdot 7^2 + 6 \cdot 3^2 + 6 \cdot 1^2 \\
\delta M_1\left(\Gamma\left(\mathbb{Z}_{2\cdot 3^2}\right)\right) &= 942
\end{aligned}$$

Subsequently, by utilizing Theorem 3.3, Equation 19, we obtain

$$\begin{aligned}
\delta M_1\left(\Gamma\left(\mathbb{Z}_{2\cdot 3^2}\right)\right) &= 5p^{2k+1} + 3p^{2k} - 5p^{k+2} - 4p^{k+1} - 5p^k + p^3 + 5p \\
&= 5 \cdot 3^{2\cdot 2+1} + 3 \cdot 3^{2\cdot 2} - 5 \cdot 3^{2+2} - 4 \cdot 3^{2+1} - 5 \cdot 3^2 + 3^3 + 5 \cdot 3 \\
&= 942.
\end{aligned}$$

According to the preceding computations, both the definition and the theorem provide identical values for the delta first Zagreb index.

Theorem 3.4. The delta second Zagreb index of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is

$$\begin{aligned}
\delta M_2\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= 7p^{2k+2} + 5p^{2k+1} - 10p^{k+3} - 7p^{k+2} - 5p^{k+1} - 2p^k \\
&\quad + \frac{7}{2}p^4 + \frac{5}{2}p^3 + \frac{7}{2}p^2 + \frac{3}{2}p + 1.
\end{aligned} \quad (21)$$

Proof. Every pair of different vertices in the set V_1, namely $u_1 \neq v_1 \in V_1$, are adjacent. Consequently, the quantity of edges linking a pair of vertices in V_1 is given by $C(|V_1|, 2) = C(p+1, 2) = \frac{1}{2}p(p+1)$. Moreover, the quantity of edges linking a vertex in V_1 to a vertex in V_2 is given by $|V_1| \times |V_2| = (p+1)(p-1)$. Likewise, the quantity of edges linking a vertex in V_1 to a vertex in V_3, and a vertex in V_1 to a vertex in V_4, are $|V_1| \times |V_3| = (p+1)(p^k - p)$ and $|V_1| \times |V_4| = (p+1)(p^k - p)$, respectively. The total number of edges linking the vertices in V_2 to those in V_3 is given by $|V_2| \times |V_3| = (p-1)(p^k - p)$. By applying the information from Theorem 3.3 and including $u_1 \in V_1$ and $u_2 \in V_2$, we obtain

$$\begin{aligned}
\delta M_2\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= \sum_{uv \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \delta_u \cdot \delta_v \\
&= \sum_{u_1v_1 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \delta_{u_1}\delta_{v_1} + \sum_{u_1v_2 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \delta_{u_1}\delta_{v_2} \\
&\quad + \sum_{u_1v_3 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \delta_{u_1}\delta_{v_3} + \sum_{u_1v_4 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \delta_{u_1}\delta_{v_4} + \\
&\quad \sum_{u_2v_3 \in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \delta_{u_2}\delta_{v_3} \\
&= \frac{1}{2}p(p+1)\left(2p^k - p - 1\right)\left(2p^k - p - 1\right) + \\
&\quad (p+1)(p-1)\left(2p^k - p - 1\right)\left(p^k - p + 1\right) + \\
&\quad (p+1)\left(p^k - p\right)\left(2p^k - p - 1\right) \cdot p + \\
&\quad (p+1)\left(p^k - p\right)\left(2p^k - p - 1\right) \cdot 1 + \\
&\quad (p-1)\left(p^k - p\right)\left(p^k - p + 1\right) \cdot p \\
\delta M_2\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= 7p^{2k+2} + 5p^{2k+1} - 10p^{k+3} - 7p^{k+2} - 5p^{k+1} - 2p^k + \frac{7}{2}p^4 + \\
&\quad \frac{5}{2}p^3 + \frac{7}{2}p^2 + \frac{3}{2}p + 1.
\end{aligned}$$

Hence, the general formula of the delta first Zagreb index of $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is as follows.

$$\begin{aligned}
\delta M_2\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= 7p^{2k+2} + 5p^{2k+1} - 10p^{k+3} - 7p^{k+2} - 5p^{k+1} - 2p^k \\
&\quad + \frac{7}{2}p^4 + \frac{5}{2}p^3 + \frac{7}{2}p^2 + \frac{3}{2}p + 1.
\end{aligned}$$

To compute the delta second Zagreb index according to its definition, it will be divided into four subsets: $V_1 = \{0, 6, 9, 12\}$, $V_2 = \{3, 15\}, V_3 = \{2, 4, 8, 10, 14, 16\}$, dan $V_4 = \{1, 5, 7, 11, 13, 17\}$. Upon examining Figure 1, it is evident that the vertices in V_1 are mutually neighboring. Moreover, every vertex in V_2 is adjacent to all other vertices. The vertex at V_2 is adjacent to the vertex V_3. Consequently, we derive

$$\begin{aligned}
\delta M_2\left(\Gamma\left(\mathbb{Z}_{2\cdot 3^2}\right)\right) &= 6 \cdot \delta_0\delta_6 + 8 \cdot \delta_0\delta_3 + 24 \cdot \delta_0\delta_2 + 24 \cdot \delta_0\delta_1 + 12 \cdot \delta_3\delta_2 \\
&= 6 \cdot 14 \cdot 14 + 8 \cdot 14 \cdot 7 + 24 \cdot 14 \cdot 3 + 24 \cdot 14 \cdot 1 + 12 \cdot 7 \cdot 3 \\
\delta M_2\left(\Gamma\left(\mathbb{Z}_{2\cdot 3^2}\right)\right) &= 3.556.
\end{aligned}$$

By applying Theorem 3.4, Equation 21, the value of the delta second Zagreb index is derived as follows.

$$\begin{aligned}\delta M_2\left(\Gamma\left(\mathbb{Z}_{2\cdot 3^2}\right)\right) &= 7p^{2k+2} + 5p^{2k+1} - 10p^{k+3} - 7p^{k+2} - 5p^{k+1} - 2p^k + \frac{5}{2}p^4 + \frac{5}{2}p^3 \\ &\quad + \frac{5}{2}p^2 + \frac{3}{2}p + 1 \\ &= 7\cdot 3^{2\cdot 2+2} + 5\cdot 3^{2\cdot 2+1} - 10\cdot 3^{2+3} - 7\cdot 3^{2+2} - 5\cdot 3^{2+1} - 2\cdot 3^2 \\ &\quad + \frac{7}{2}\cdot 3^4 + \frac{5}{2}\cdot 3^3 + \frac{7}{2}\cdot 3^2 + \frac{3}{2}\cdot 3 + 1 \\ &= 3.556\end{aligned}$$

Theorem 3.5. The delta hyper Zagreb index of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is

$$\begin{aligned}\delta HM\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= 9p^{3k+1} + 7p^{3k} + 4p^{2k+2} - 5p^{2k+1} - 15p^{2k} \\ &\quad - 15p^{k+3} - 9p^{k+2} + 12p^{k+1} \\ &+5p^4 + 5p^3 + 2p.\end{aligned} \tag{22}$$

Proof. Using the same assumptions and information as in the proof of Theorem 3.4, we obtain

$$\begin{aligned}\delta HM\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= \sum_{uv\in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_u + \delta_v\right)^2 \\ &= \sum_{u_1v_1\in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{u_1} + \delta_{v_1}\right)^2 + \sum_{u_1v_2\in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{u_1} + \delta_{v_2}\right)^2 \\ &+ \sum_{u_1v_3\in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{u_1} + \delta_{v_3}\right)^2 + \sum_{u_1v_4\in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{u_1} + \delta_{v_4}\right)^2 \\ &+ \sum_{u_2v_3\in E\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{u_2} + \delta_{v_3}\right)^2 \\ &= \frac{1}{2}p\,(p+1)\left(\left(2p^k - p - 1\right) + \left(2p^k - p - 1\right)\right)^2 + (p+1)(p-1) \\ &\quad \left(\left(2p^k - p - 1\right) + \left(p^k - p + 1\right)\right)^2 + (p+1)\left(p^k - p\right) \\ &\quad \left(\left(2p^k - p - 1\right) + p\right)^2 + (p+1)\left(p^k - p\right)\left(\left(2p^k - p - 1\right) + 1\right)^2 + \\ &\quad (p-1)\left(p^k - p\right)\left(\left(p^k - p + 1\right) + p\right)^2 \\ \delta HM\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= 9p^{3k+1} + 7p^{3k} + 4p^{2k+2} - 5p^{2k+1} - 15p^{2k} - 15p^{k+3} - 9p^{k+2} + \\ &\quad 12p^{k+1} + 5p^4 + 5p^3 + 2p\end{aligned}$$

The general formula of the delta hyper Zagreb index of $\Gamma\left(\mathbb{Z}_{p^k}\right)$ is as follows

$$\begin{aligned}\delta HM\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= 9p^{3k+1} + 7p^{3k} + 4p^{2k+2} - 5p^{2k+1} - 15p^{2k} - 15p^{k+3} - 9p^{k+2} + \\ &\quad 12p^{k+1} + 5p^4 + 5p^3 + 2p\end{aligned}$$

Theorem 3.6. The delta forgotten index of $\Gamma\left(\mathbb{Z}_{2p^k}\right)$ is

$$\begin{aligned}\delta F\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= 9p^{3k+1} + 7p^{3k} + 10p^{k+3} + 9p^{k+2} + 27p^{k+1} + 4p^k \\ &\quad - 15p^{2k+2} - 18p^{2k+1} - 15p^{2k} - 3p^4 - 12p^2 - p - 2.\end{aligned} \tag{23}$$

Proof Utilizing the provided facts in the proof of Theorem 3.3, we obtain

$$\begin{aligned}\delta F\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right) &= \sum_{v\in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_v\right)^3 \\ &= \sum_{v_1\in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_1}\right)^3 + \sum_{v_2\in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_2}\right)^3 + \\ &\quad \sum_{v_3\in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_3}\right)^3 + \sum_{v_4\in V\left(\Gamma\left(\mathbb{Z}_{2p^k}\right)\right)} \left(\delta_{v_4}\right)^3 \\ &= (p+1)\left(2p^k - p - 1\right)^3 + (p-1)\left(p^k - p + 1\right)^3 + \\ &\quad \left(p^k - p\right)(p)^3 + \left(p^k - p\right)(1)^3 \\ \delta F\left(\Gamma\left(\mathbb{Z}_{p^k}\right)\right) &= 9p^{3k+1} + 7p^{3k} + 10p^{k+3} + 9p^{k+2} + 27p^{k+1} + 4p^k - \\ &\quad - 15p^{2k+2} - 18p^{2k+1} - 15p^{2k} - 3p^4 - 12p^2 - p - 2\end{aligned}$$

According to the findings from Theorems 3.3, 3.4, 3.5, and 3.6, we conduct an additional analysis employing statistical methods as follows.

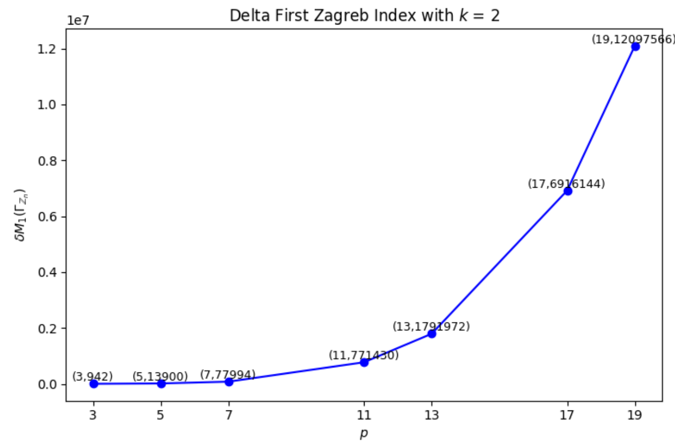

Figure 2. Graphics of Delta First Zagreb Index with $k = 2$

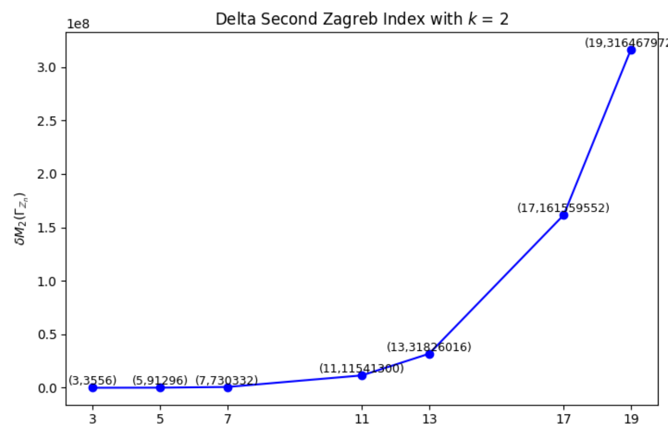

Figure 3. Graphics of Delta Second Zagreb Index with $k = 2$

Table 2. Results of Comparison Test of Index Value Distribution for k = 2 and k = 3 with Levene's Test

Variables	k	Significance Level	Result
delta first Zagreb >< delta second Zagreb	2	0.021	unequal
	3	0.046	unequal
delta first Zagreb >< delta forgotten	2	0.027	unequal
	3	0.035	unequal
delta first Zagreb >< delta hyper Zagreb	2	0.025	unequal
	3	0.035	unequal
delta second Zagreb >< delta forgotten	2	0.034	unequal
	3	0.035	unequal
delta second Zagreb >< delta hyper Zagreb	2	0.031	unequal
	3	0.035	unequal
delta forgotten >< delta hyper Zagreb	2	0.766	equal
	3	0.983	equal

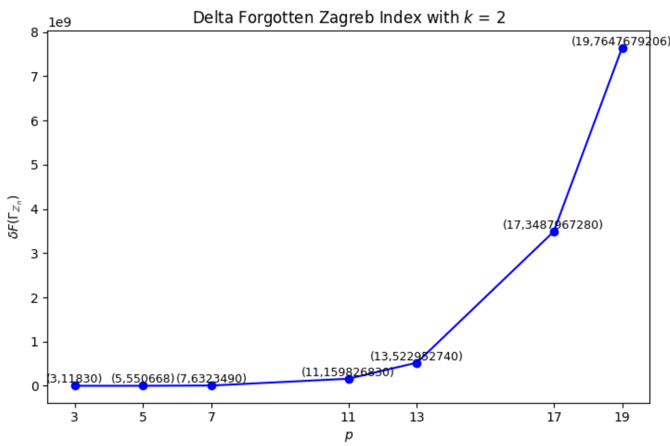

Figure 4. Graphics of Forgotten Index with k = 2

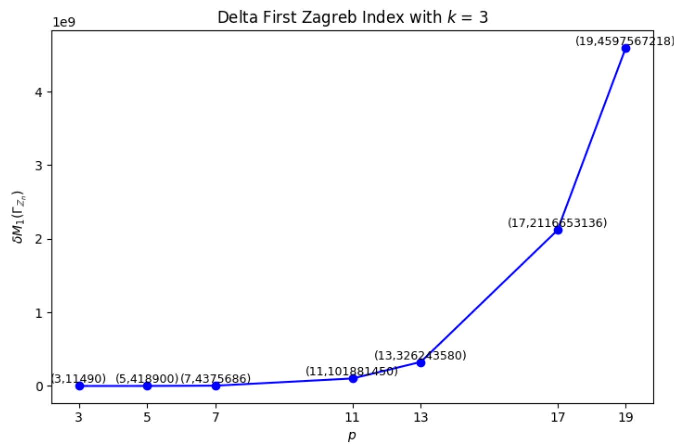

Figure 6. Graphics of Delta First Zagreb Index with k = 3

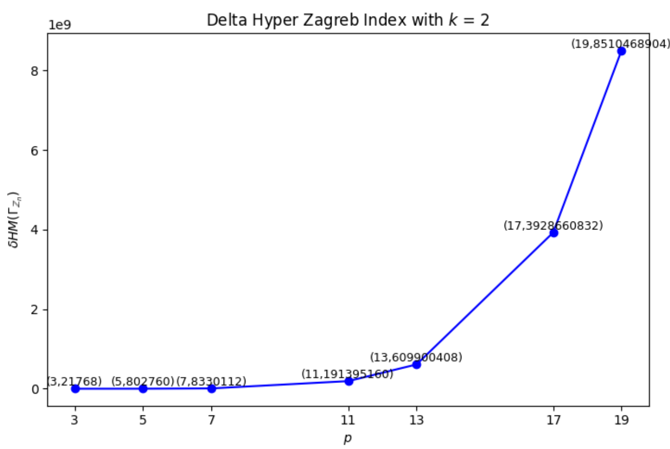

Figure 5. Graphics of Hyper Zagreb Index with k = 2

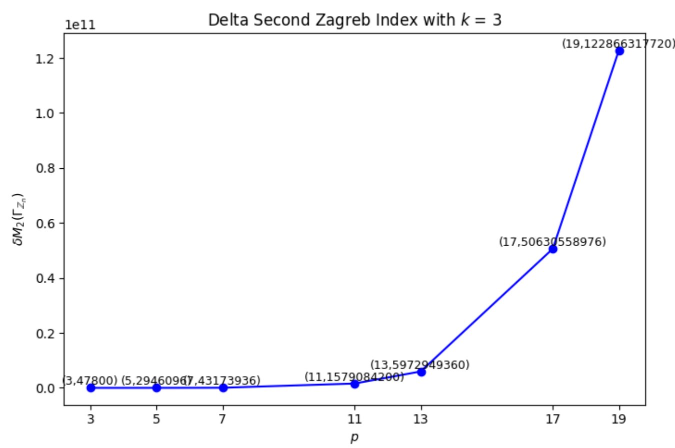

Figure 7. Graphics of Delta Second Zagreb Index with k = 3

Figures 2, 3, 4, 5 illustrate the graph trajectory for each index. At k = 2, delta hyper Zagreb exhibits superior movement relative to other indices. Upon closer examination, the delta forgotten index value bears a striking resemblance to the delta hyper Zagreb. The delta first Zagreb exhibits a diminished value compared to the other three indices for the identical p

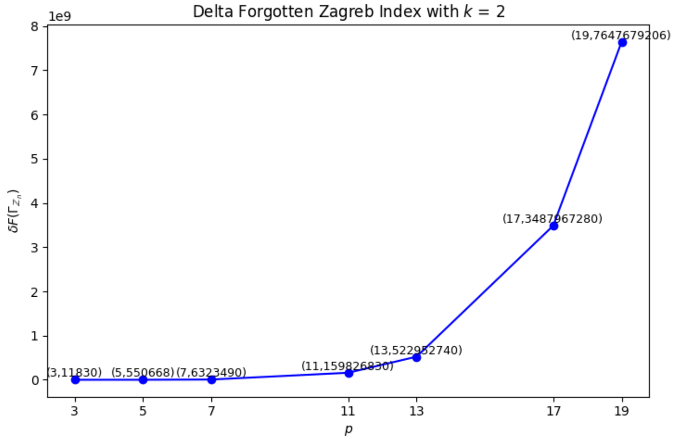

Figure 8. Graphics of Forgotten Index with k = 3

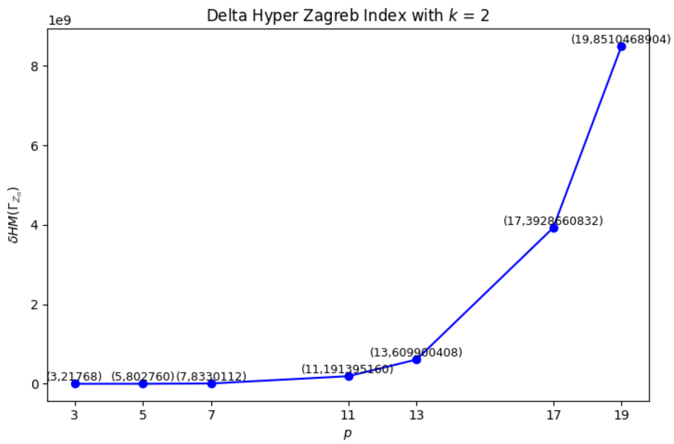

Figure 9. Graphics of Hyper Zagreb Index with k = 3

and k parameters. The assertion of the index value at $k = 2$ is corroborated in Figures 6, 7, 8, and 9 by the index value at $k = 3$. Despite the significant increase in the value of each index relative to $k = 2$, delta hyper Zagreb maintains a superior value compared to the other indices, while it is not markedly distant from the value of the forgotten index. A comparison analysis of data distribution was conducted using the Levene test to see whether the two indices exhibit similar value distributions.

Table 2 elucidates the outcomes of the comparative analysis of distribution among indices with a margin of error of 5%. With a significance value exceeding 5%, the forgotten index has a distribution analogous to the hyper Zagreb only at both $k = 2$ and $k = 3$. The significant value rises from $k = 2$ to $k = 3$, indicating that the similarity of the forgotten index data distribution to the hyper Zagreb hyper index becomes more pronounced as the k value grows from 2 to 3. Comparisons between other indices yield a significant value below the 5% error threshold, indicating a substantial difference between the two indices being compared.

4. CONCLUSIONS

In this paper, we have investigated delta-based topological indices for the prime coprime graph associated with the integers modulo group $\mathbb{Z}_n$. Explicit formulas were derived for the delta first and second Zagreb indices, the delta hyper-Zagreb index, and the delta forgotten index. We further supported these results with systematic computations and tabulated data for representative values of n. In addition, a comparative statistical analysis was conducted using the Levene test, which provided insights into the distributional similarities and differences among the indices.

The present study opens several avenues for further exploration. One natural direction is to extend the investigation to non-Abelian groups and to explore spectral properties of prime coprime graphs under delta degree modifications. Another promising extension involves generalizing the current framework to advanced graph structures such as fuzzy graphs and intuitionistic fuzzy graphs, where indices like the Sombor and Zagreb indices have already been studied (Arif et al., 2023). Similarly, delta-based indices could be examined within neutrosophic graphs (Fujita and Smarandache, 2025a) and hypergraphs (or superhypergraphs) (Fujita and Smarandache, 2025b,c), where structural complexity provides new challenges and opportunities, representing a significant direction for future work.

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