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Research Paper

Some Properties of Generalized Token Graphs

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Abstract
The generalized k-token graph $GF_k(G)$ is a graph with the k-subsets of $V(G)$ as the vertices, and two different vertices are adjacent if and only if the symmetric difference contains at least one edge of G. This model extends the classical k-token graph by relaxing the adjacency condition, leading to increased edge density and altered topological properties. In this paper, we establish the fundamental properties of $GF_k(G)$, including its connectivity, duality, and monotonicity. We provide exact formulas for the vertex degrees and the total size of $GF_2(G)$, along with combinatorial bounds for $k > 2$. Furthermore, we characterize the girth and clique numbers, proving that $GF_k(G)$ is highly prone to containing triangles even when the base graph is triangle-free. We also explore the inheritance of Hamiltonicity and bipartiteness, demonstrating that while connectivity is preserved, bipartiteness is lost for almost all bipartite graphs with at least four vertices. Our results provide a comprehensive structural characterization of this generalization, bridging the gap between classical token graphs and broader set-theoretic graph constructions.

Keywords
Graph, Token Graph, Generalized Token Graph

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1. INTRODUCTION

The study of configurations on graphs has led to the development of token graphs, a field that traces its origins back to 1991. Alavi et al. (1991) first introduced the double vertex graph, where vertices are 2-element subsets of $V(G)$ and adjacency is defined by the symmetric difference of two distinct subsets being an edge in $E(G)$. This concept was later expanded by Audenaert et al. (2002) to subsets of cardinality k, resulting in the symmetric power graph, which has proven instrumental in addressing graph isomorphism problems and finding applications within quantum mechanics. General foundational theories regarding such graph structures are well-documented in classic literature (West, 2001; Bondy and Murty, 2008).

A pivotal advancement occurred when Fabila-Monroy et al. (2012) established a formal connection between symmetric power graphs and the graph pebbling problem. In this context, vertices correspond to configurations of k indistinguishable tokens distributed over the vertices of G, and adjacency corresponds to the movement of a single token along an edge. The concept of "token movement" has established token graphs as a key subject in numerous fields, including distributed memory allocation, robotic pathfinding, and deflection routing (Auletta et al., 1999). Beyond these applications, token graphs have invited rigorous algebraic scrutiny, particularly regarding their automorphisms (Susanti, 2023; Zhang et al., 2023), Laplacian spectra (Dalfó et al., 2021; Brouwer and Haemers, 2011), and edge transitivity (Zhang et al., 2023). Recent studies have further explored their algebraic connectivity for infinite families of graphs (Dalfó and Fiol, 2024), the application of Garland's method (Lew, 2024), and the introduction of token signed graphs to measure the unbalance level of a graph (Dalfó et al., 2025; Godsil and Royle, 2013). The edge-connectivity and vertex-connectivity of token graphs, particularly for trees and general graphs, have been thoroughly characterized (Leaños and Ndjatchi, 2021; Fabila-Monroy et al., 2022). Similarly, the existence of Hamiltonian cycles in the token graphs of specific graph classes, such as join graphs and fan graphs, has been deeply investigated (Adame et al., 2021; Mirajkar and Priyanka, 2016; Rivera and Trujillo-Negrete, 2021; Rivera Martínez and Trujillo-Negrete, 2018). In a similar vein, recent studies have further explored the structural properties of 3-token graphs for disjoint union of graphs and some specific graph families (Djuang et al., 2025). The edge-connectivity and vertex-connectivity of token graphs, particularly for trees and general graphs, have been thoroughly characterized (Leaños and Ndjatchi, 2021; Fabila-Monroy et al., 2022).

Formally, let $X_k(G)$ denote the collection of all k-element subsets of $V(G)$. The k-token graph, denoted by $F_k(G)$, takes

$X_k(G)$ as its vertex set. Two vertices A and B are considered adjacent provided that their symmetric difference $A \triangle B$ equals an edge C, where $C \in E(G)$. Crucially, this definition implies that if $|A \triangle B| \neq 2$, the vertices A and B cannot be adjacent. While the symmetric difference of two distinct k-subsets is always even (as noted in Lemma 3.1), the classical definition restricts adjacency strictly to the minimum possible difference of exactly two elements.

The classical k-token graph effectively models the configuration space of tokens where adjacency captures a single, elementary move along an edge. While this "unit-step" constraint is suitable for modeling minimal local transitions, it may be too rigid for many real-world and theoretical systems. In many scenarios, two configurations may differ by more than one elementary change yet still share a fundamental structural link induced by the underlying graph G. In domains such as information theory or network security, two states are often considered "related" or "accessible" if they share at least one valid interaction link, regardless of how many other components have changed state simultaneously.

Efforts to generalize this concept have recently appeared in the literature. For instance, Herrera-Ramirez and Hoekstra Mendoza (2025) proposed a generalization where adjacency is defined by the simultaneous movement of m tokens along m edges of G. While their model expands the reachability of configurations, it still relies on a fixed number of discrete token displacements. In contrast, our approach shifts the focus from the quantity of movement to an abstract configuration relation anchored by the existence of a structural link. By relaxing the adjacency condition, we allow the model to capture these higher-order interactions while still preserving the "locality" anchored by the edges of G.

Accordingly, we generalize the definition of token graphs by weakening the adjacency requirement from $C = A \triangle B$ to $C \subseteq A \triangle B$. Under this condition, two distinct vertices A and B may be adjacent regardless of the cardinality of their symmetric difference, provided it contains at least an edge in G.

Definition 1.1. *For a given graph G and an integer $k \geq 1$, the generalized k-token graph of G, denoted by $GF_k(G)$, is a graph with vertex set $X_k(G)$ and for every $A, B \in X_k(G)$, $\{A, B\} \in E(GF_k(G))$ if and only if $A \triangle B \supseteq C$ for some $C \in E(G)$.*

This definition ensures that $F_k(G)$ is always a spanning subgraph of $GF_k(G)$, effectively "filling in" the configuration space with additional edges that represent more complex transitions. A related idea of generalization appeared in Huda and Susanti (2025), though it utilized intersection operations rather than symmetric differences. The objective of this paper is to investigate how this added edge density affects the global structural properties of $GF_k(G)$. We provide a rigorous graph-theoretical analysis of its basic isomorphisms, girth, Hamiltonicity, and bipartiteness, establishing a foundation for understanding these dense configuration spaces. For clarity, we present some examples of the graph below. Figure 1 shows the generalized 2-token graph of path graph P_4 with vertex set $\{0, 1, 2, 3\}$ and edge set $\{\{0, 1\}, \{1, 2\}, \{2, 3\}\}$, while Figure 2 represents the generalized 3-token graph of cycle graph C_5 with vertex set $\{0, 1, 2, 3, 4\}$ and edge set $\{\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 0\}\}$.

2. EXPERIMENTAL SECTION

We begin by establishing the necessary graph theoretic notations and defining the standard token graph structure. Throughout this paper, all graphs are considered to be finite, undirected, and simple, meaning they contain no directed edges, multiple edges, or loops. Let $G = (V(G), E(G))$ be a graph where $V(G)$ is the vertex set and $E(G)$ is the edge set. The order and size of G represent the number of vertices and edges of G, respectively. Since G is simple, an edge is represented by a 2-element subset of $V(G)$. Two edges S_1 and S_2 of G are said to be disjoint if $S_1 \cap S_2 = \emptyset$. Regarding specific graph classes, G is called a complete graph of order n, denoted by K_n, if any two distinct vertices are joined by an edge. Conversely, G is called a null graph of order n, denoted by N_n, if its size is zero.

We now introduce some basic substructures and vertex properties of graphs. We define a graph H as a subgraph of G provided that its vertex and edge set is a subset of those in G, denoted by $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Specifically, where H shares the identical vertex set (i.e. $V(H) = V(G)$), it is termed a spanning subgraph. For a given subset $W \subseteq V(G)$, H is called a subgraph of G induced by W if $V(H) = W$ and for all $x, y \in V(H)$, $\{x, y\} \in E(G)$ implies $\{x, y\} \in E(H)$. The degree of a vertex v in G, denoted by $\deg_G(v)$, is the number of vertices adjacent to v. A graph G is said to be r- regular if every vertex $v \in V(G)$ satisfies $\deg_G(v) = r$.

A graph G is called a bipartite graph if $V(G)$ can be partitioned into two subsets X and Y such that $E(G) \subseteq \{\{x, y\} : x \in X, y \in Y\}$. We denote such a graph by $G[X, Y]$, and the partition $\{X, Y\}$ is called a bipartition of the graph, where X and Y are its parts. A graph $G[X, Y]$ is called a complete bipartite graph if $E(G) = \{\{x, y\} : x \in X, y \in Y\}$. For instance, the star graph $K_{1,n-1}$ is a complete bipartite graph $G[X, Y]$ with $|X| = 1$ or $|Y| = n - 1$.

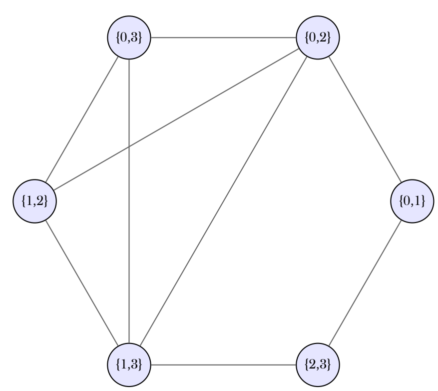

Figure 1. The Generalized 2-Token Graph of Path Graph P_4

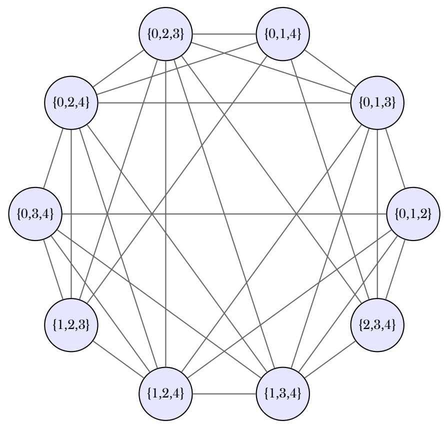

Figure 2. The Generalized 3-Token Graph of Cycle Graph C_5

For $m \leq |V(G)|$, we denote $B = (u_i)_{i=1}^m$ as a finite sequence of adjacent vertices in G where u_i and u_{i+1} are adjacent for all $1 \leq i \leq m-1$. The sequence is said to join u_i and u_j for any $i, j \in \{1, 2, \ldots, m\}$. Consider the sequence B: if $u_i \neq u_j$ for any $i \neq j$, then B is called a path of G. Specifically, B is called a cycle of G if B forms a path with the exception that $u_1 = u_m$ for $m \geq 4$. A path and a cycle can be viewed as subgraphs that induce a path graph P_m and a cycle graph C_{m-1} (for $m \geq 4$) of G, respectively. If a path (or cycle) B joins all vertices of G, then B is called a Hamiltonian path (or Hamiltonian cycle). Consequently, G is called a Hamiltonian graph if it contains a Hamiltonian cycle. A graph G is connected if any two distinct vertices are joined by at least one path; otherwise, it is disconnected. The length of a path (or cycle) represents the number of edges joining its vertices. The girth of G, denoted by girth(G), is defined as the minimum length of a cycle in G. Furthermore, G is said to be triangle-free if it does not contain a cycle of length 3. A graph is called a tree if it is connected but contains no cycles.

Let G_1 and G_2 be two graphs. If there exists an injective function $\theta : V(G_1) \to V(G_2)$ such that for any $u, v \in V(G_1)$, $\{u, v\} \in E(G_1)$ implies $\{\theta(u), \theta(v)\} \in E(G_2)$, then θ is called an injective homomorphism and G_1 is isomorphic to a subgraph of G_2. Specifically, G_1 and G_2 are isomorphic, denoted by $G_1 \cong G_2$, if such θ is a bijective function and for any $u, v \in V(G_1)$, $\{u, v\} \in E(G_1)$ if and only if $\{\theta(u), \theta(v)\} \in E(G_2)$. Let G_1 and G_2 be two disjoint graphs. We define their union, denoted as $G_1 \cup G_2$, by the vertex set $V(G_1) \cup V(G_2)$ and the edge set $E(G_1) \cup E(G_2)$. Furthermore, the Cartesian product $G_1 \square G_2$ is constructed with the vertex set $V(G_1) \times V(G_2)$. In this product graph, two vertices (u, v) and (w, z) are adjacent if and only if either $u = w$ and $\{v, z\} \in E(G_2)$, or if $v = z$ and $\{u, w\} \in E(G_1)$.

Unless otherwise specified, we assume G represents a graph of order n and k is a positive integer. Some results from related research that will be used in this paper are listed below.

Theorem 2.1. (Fabila-Monroy et al., 2012) *Let G be a connected graph with diameter δ. Then $F_k(G)$ is connected with diameter at least $k(\delta - k + 1)$ and at most $k\delta$.*

Proposition 2.2. (Fabila-Monroy et al., 2012) $|E(F_k(G))| = \binom{n-2}{k-1}|E(G)|$.

Theorem 2.3. (Deepalakshmi et al., 2020) *$F_2(G)$ is a tree if and only if $G = P_2$ or P_3.*

Theorem 2.4. (Deepalakshmi et al., 2020) *If there exists a vertex $v_1 \in V(G)$ such that $\deg(v_1) = 2$, then G isomorphic to a subgraph of $F_2(G)$.*

Theorem 2.5. (Deepalakshmi et al., 2020) *G is triangle free if and only if $F_2(G)$ is triangle free.*

Theorem 2.6. (Deepalakshmi et al., 2020) *$F_2(G)$ is isomorphic to the line graph $L(G)$ if and only if $G = K_n$.*

Theorem 2.7. (Mirajkar and Priyanka, 2016) *$F_k(G)$ is a Hamiltonian graph if and only if G is a complete graph.*

Theorem 2.8. (Deepalakshmi et al., 2020) *Let G be a connected graph. $F_2(G)$ is bipartite if and only if G is bipartite.*

Theorem 2.9. (Fabila-Monroy et al., 2012) *If a graph G contains a Hamiltonian path and n is even and k is odd, then $F_k(G)$ contains a Hamiltonian path.*

Theorem 2.10. (Fabila-Monroy et al., 2012) *If $H_1, \ldots, H_m$ are pairwise disjoint induced subgraphs of G, then for all integers $s_1, \ldots, s_m$ such that $1 \leq s_i \leq |V(H_i)|$ and $\sum_{i=1}^m s_i = k$, it follows that $F_{s_1}(H_1) \square \ldots \square F_{s_m}(H_m)$ is an induced subgraph of $F_k(G)$.*

Based on established results in token graphs, we conducted an investigation to extend these concepts to generalized token graphs. We systematically examined whether existing theorems and lemmas remain valid or require additional constraints in this generalized setting. Finally, we formulated conjectures regarding these properties to provide directions for future research.

3. RESULTS AND DISCUSSIONS

We begin by establishing a fundamental property of the symmetric difference between two sets, which serves as the basis for the adjacency definition in generalized token graphs.

Lemma 3.1. *Consider arbitrary distinct sets A and B, each having k elements. We observe that*

$$|A \Delta B| \in \{2i : 1 \leq i \leq k\}.$$

Proof. Let A and B be two non-identical sets, each with cardinality k. Consequently, the intersection size satisfies $1 \leq k - |A \cap B| \leq k$. By applying standard set theory identities, we observe that

$$\begin{aligned} |A \Delta B| &= |A \cup B| - |A \cap B| \\ &= |A| + |B| - 2|A \cap B| \\ |A \Delta B| &= 2(k - |A \cap B|) \end{aligned}$$

Thus, the number of elements of $|A \Delta B|$ is always even and satisfies $2 \leq |A \Delta B| \leq 2k$.

The following theorem formalizes the observation that the generalized token graph, by virtue of its adjacency rule, acts as a natural extension of the classical model while maintaining connectivity.

Theorem 3.2. *Let G be a graph.*
(i) *$F_k(G)$ is a spanning subgraph of $GF_k(G)$.*
(ii) *If G is connected, then $GF_k(G)$ is connected.*

Proof. (i) By definition, $V(F_k(G)) = V(GF_k(G)) = X_k(G)$. Let A, B be adjacent vertices in $F_k(G)$. Then $A \triangle B = \{u, v\}$ for some edge $\{u, v\} \in E(G)$. Since $\{u, v\} \subseteq A \triangle B$, it satisfies the adjacency condition for $GF_k(G)$ given in Definition 1.1. Thus, $E(F_k(G)) \subseteq E(GF_k(G))$.
(ii) If G is connected, then by Theorem 2.1, $F_k(G)$ is connected. Since $F_k(G)$ is a spanning subgraph of $GF_k(G)$, adding edges to a connected graph preserves connectivity. Therefore, $GF_k(G)$ is connected.

Remark 3.3. *The converse of Theorem 3.2 (ii) is not always true. For instance, let $G \cong K_2 \cup K_1$ be a disconnected graph. Its generalized token graph $GF_2(G)$ is isomorphic to P_3, which is connected.*

We now address the trivial cases of generalized token graphs. The following theorem characterizes the graph for extremal values of k and establishes a duality property based on set complementation.

Theorem 3.4. *Let G be a graph of order n. The following isomorphisms hold:*
(i) *$GF_k(G) \cong GF_{n-k}(G)$ for any $1 \le k < n$.*
(ii) *$GF_1(G) \cong G \cong GF_{n-1}(G)$.*

Proof. (i) Define a bijection f from $X_k(G)$ to $X_{n-k}(G)$ with $f(A) = V(G) \setminus A$, for all $A \in X_k(G)$. It can be seen that for all $A, B \in X_k(G)$, $\{A, B\} \in E(GF_k(G))$ if and only if there exists $\{x, y\} \in E(G)$ such that $\{x, y\} \subseteq A \triangle B = (V(G) \setminus A) \triangle (V(G) \setminus B)$ if and only if $\{f(A), f(B)\} \in E(GF_{n-k}(G))$. Therefore, f is an isomorphism.
(ii) The isomorphism $GF_1(G) \cong GF_{n-1}(G)$ follows immediately from result (i) by setting $k = 1$. Next, we show that $G \cong GF_1(G)$. Define a bijection f from $V(G)$ to $X_1(G)$ with $f(x) = \{x\}$, for all $x \in V(G)$. Consider for any $\{x\}, \{y\} \in X_1(G)$ with $x \neq y$, we have $\{x\} \triangle \{y\} = \{x, y\}$. Thus, for any $x, y \in V(G)$, we obtain $\{x, y\} \in E(G)$ if and only if $\{f(x), f(y)\} = \{\{x\}, \{y\}\} \in E(GF_1(G))$.

Consider a graph G. It is evident that the order of $GF_n(G)$ is 1. Furthermore, if G is an edgeless graph (i.e., $|E(G)| = 0$), then by Definition 1.1, $GF_k(G)$ is also edgeless. To avoid these trivial cases, we assume for further discussion that $2 \le k \le n-2$ and $|E(G)| \ge 1$.

Recall that the degree of a vertex A in $GF_k(G)$, denoted by $\deg_{GF_k(G)}(A)$, is the number of vertices adjacent to A. By Definition 1.1, a vertex $B \in X_k(G) - \{A\}$ is adjacent to A if and only if $A \triangle B$ contains an edge of G. In set-theoretic terms, this adjacency condition implies that the collection of all subsets of $A \triangle B$, denoted by $2^{A \triangle B}$, must share at least one element with the edge set $E(G)$. Therefore, the degree of A can be expressed as the count of such vertices B:

$$\deg_{GF_k(G)}(A) = \sum_{\substack{B \in X_k(G) - \{A\} \\ 2^{A \triangle B} \cap E(G) \neq \emptyset}} 1.$$

Specifically for $k = 2$, we obtain the exact formula for the degree of any vertex of $GF_2(G)$ as follows.

Theorem 3.5. *For any $A = \{x, y\} \in X_2(G)$, $\deg_{GF_2(G)}(A)$ is $\binom{n-2}{2} + \deg_G(x) + \deg_G(y) - 2$ if $\{x, y\} \in E(G)$ and $\deg_G(x) + \deg_G(y) + \sum_{\substack{a,b \in V(G) - \{x,y\} \\ 2^{\{x,y,a,b\}} \cap E(G) \neq \emptyset}} 1$ if $\{x, y\} \notin E(G)$.*

Proof. Consider the following cases.
1. If $\{x, y\} \in E(G)$, then there are three possibilities.
   (a) We observe that $\{x, y\}$ is adjacent to any vertex $\{a, b\}$ provided $\{a, b\} \neq \{x, y\}$. There are precisely $\binom{n-2}{2}$ such vertices available.
   (b) The vertex $\{x, y\}$ is also adjacent to any set $\{x, a\}$ where $a \neq x$. The number of such vertices corresponds to the number of neighbors of x within $V(G) \setminus \{y\}$, which equals $\deg_G(x) - 1$.
   (c) The vertex $\{x, y\}$ is adjacent to all vertices $\{y, b\}$ such that $b \neq y$. The number of $\{y, b\}$ is calculated by the number of vertices adjacent to y in $V(G) \setminus \{x\}$, that is $\deg_G(y) - 1$.

   Therefore, in this case, $\deg_{GF_2(G)}(\{x, y\}) = \binom{n-2}{2} + \deg_G(x) + \deg_G(y) - 2$.
2. If $\{x, y\} \notin E(G)$, then there are three possibilities.
   (a) The vertex $\{x, y\}$ is adjacent to any vertex $\{x, a\}$ where $a \neq x$. The number of $\{x, a\}$ is calculated by the number of vertices adjacent to x in $V(G)$, that is $\deg_G(x)$.
   (b) The vertex $\{x, y\}$ is also adjacent to all vertices $\{y, b\}$ where $b \neq y$. The number of $\{y, b\}$ is calculated by the number of vertices adjacent to y in $V(G)$, that is $\deg_G(y)$.
   (c) $\{x, y\}$ is adjacent to all vertices $\{a, b\}$ where $\{x, y\} \cap \{a, b\} = \emptyset$. We have either $\{x, a\}$, $\{y, a\}$, $\{x, b\}$, $\{y, b\}$, or $\{a, b\}$ must be the edges of $E(G)$. Clearly, the number of all of them can be represented as

$$\sum_{\substack{a,b \in V(G) - \{x,y\} \\ 2^{\{x,y,a,b\}} \cap E(G) \neq \emptyset}} 1.$$

The authors in Huda and Lestari (2024) recently found partial results which formulate the degrees for $k > 2$, by using the inclusion-exclusion principle, Corradi's lemma, and Bonferroni's inequality. Let β be the maximum cardinality of pairwise intersection elements in $|E(G)|$ and let $[A] := \left|2^A \cap E(G)\right|$. Let

$r = \frac{p \min^2\{|E-A|:E\in E(G)\}}{\min\{|E-A|:E\in E(G)\}+(p-1)\beta}$ and $u = \min\{k,\ p\max\{|E-A| : E \in E(G)\}\}$. For any vertex A of $GF_k(G)$, it follows that $\deg_{GF_k(G)}(A)$ is less than or equal to $\sum_{p=1}^{|E(G)|}(-1)^{p-1}\sum_r^u \binom{n-r}{k-r}$ $\phi_{p,r}$ if $[A] = 0$ and is greater than or equal to $\sum_{p=1}^{[A]}(-1)^{p-1}$ $\sum_{r=\frac{4p}{2+(p-1)\beta}}^{2p} \binom{n-r}{k}\gamma_{p,r}$ if $[A] > 0$, where $\phi_{p,r}$ represents the number of subcollections $\{E_i\}_{1\leq i\leq p}$ of $E(G)$ such that $|\bigcup_{1\leq i\leq p}(E_i - A)| = r$ and $\gamma_{p,r}$ represents the number of subcollections $\{E_i\}_{1\leq i\leq p}$ of $2^A \cap E(G)$ such that $\left|\bigcup_{1\leq i\leq p} E_i\right| = r$. It will be quite challenging for the reader who interested to determine the precise formula for $\deg_{GF_k(G)}(A)$ for any $k > 2$.

With the explicit degree formula for $k = 2$ established in Theorem 3.5, we can now determine the total number of edges in the graph. By applying the Handshaking Lemma-which states that the sum of degrees is twice the number of edges-we derive the exact size of $GF_2(G)$.

Theorem 3.6. *The size of $GF_2(G)$ is*
$\frac{1}{2}\left[\binom{n-2}{2} - 2\right]|E(G)| + \sum_{v\in V(G)} \frac{n\deg_G(v)^2 - \deg_G(v)^3}{2} + \sum_{\{x,y\}\notin E(G)} \sum_{\substack{\{a,b\}\neq\{x,y\} \\ 2^{\{x,y,a,b\}}\cap E(G)\neq\emptyset}} \frac{1}{2}$.

Proof. By using Handshaking Lemma and Theorem 3.5, we have $|E(GF_2(G))|$ is equal to $\frac{1}{2}\sum_{x,y\in V(G)} \deg_{GF_2(G)}(\{x,y\})$ which is equal to

$$\frac{1}{2}[\textstyle\sum_{\{x,y\}\in E(G)} \deg_{GF_2(G)}(\{x,y\})] + \frac{1}{2}[\textstyle\sum_{\{x,y\}\notin E(G)} \deg_{GF_2(G)}(\{x,y\})] \quad (1)$$

Consider the following.

1. For $\{x,y\} \in E(G)$, $\sum_{\{x,y\}\in E(G)} \deg_{GF_2(G)}(\{x,y\})$ is equal to $\left[\binom{n-2}{2} - 2\right]m + \sum_{\{x,y\}\in E(G)} \deg_G(x) + \deg_G(y)$ which is equal to $\left[\binom{n-2}{2} - 2\right]m + \sum_{v\in V(G)} \deg_G(v)^2$ (2)
2. For $\{x,y\} \notin E(G)$, $\sum_{\{x,y\}\notin E(G)} \deg_{GF_2(G)}(\{x,y\})$ is equal to

$$\begin{aligned}
& \sum_{\{x,y\}\notin E(G)} (\deg_G(x) + \deg_G(y)) \\
& + \sum_{\{x,y\}\notin E(G)} \sum_{\substack{\{a,b\}\neq\{x,y\} \\ |2^{\{x,y,a,b\}}\cap E(G)|\neq 0}} 1 \\
= & \sum_{v\in V(G)} (n-1-\deg_G(v))\deg_G(v)^2 \\
& + \sum_{\{x,y\}\notin E(G)} \sum_{\substack{\{a,b\}\neq\{x,y\} \\ |2^{\{x,y,a,b\}}\cap E(G)|\neq 0}} 1 \\
= & (n-1)\sum_{v\in V(G)} \deg_G(v)^2 - \sum_{v\in V(G)} \deg_G(v)^3 \\
& + \sum_{\{x,y\}\notin E(G)} \sum_{\substack{\{a,b\}\neq\{x,y\} \\ |2^{\{x,y,a,b\}}\cap E(G)|\neq 0}} 1 \quad (3)
\end{aligned}$$

Substituting (2) and (3) into (1), we obtain the desired result.

While the exact size formulation is attainable for $k = 2$, the combinatorial complexity increases significantly for higher values of k. Instead of an exact enumeration, we provide bounds for the size of $GF_k(G)$ relative to the size of the underlying graph G for any arbitrary k, as follows.

Theorem 3.7.

$$\binom{n-2}{k-1} \leq \frac{|E(GF_k(G))|}{|E(G)|} \leq \left[\binom{n-2}{k-1}^2 + \binom{n-2}{k-2}\binom{n-2}{k}\right].$$

Proof. As we assume that $|E(G)| > 0$, the ratio can be determined. From Theorem 3.2 (i) and Proposition 2.2, we obtain the lower bound for $|E(GF_k(G))|$. Take any $\{x,y\} \in E(G)$. Define S_x, S_y, S_{xy}, and R be the set of vertices of $GF_k(G)$ that contain $\{x\}$, $\{y\}$, $\{x,y\}$, and neither $\{x\}$ nor $\{y\}$, respectively. In this case, the symmetric difference operation allows us to make two complete bipartite graphs $G[S_x, S_y]$ and $G[S_{xy}, R]$. Therefore, the edge $\{x,y\}$ provides

$$|S_x \backslash S_{xy}||S_y \backslash S_{xy}| + |S_{xy}||R|$$

edges for $GF_k(G)$. If $|E(G)| = 1$, then $|E(GF_k(G))|$ will be equal to

$$\begin{aligned}
& [|S_x||S_y| + |S_{xy}||R|]\,|E(G)| \\
= & \left[|S_x||S_y| + |S_{xy}|\left[\binom{n}{k} - (|S_x| + |S_y| - |S_{xy}|)\right]\right]|E(G)| \\
= & \left[\binom{n-2}{k-1}^2 + \binom{n-2}{k-2}\left[\binom{n}{k} - 2\binom{n-1}{k-1} + \binom{n-2}{k-2}\right]\right]|E(G)| \\
= & \left[\binom{n-2}{k-1}^2 + \binom{n-2}{k-2}\left[\binom{n}{k} - \binom{n-1}{k-1} - \binom{n-2}{k-1}\right]\right]|E(G)| \\
= & \left[\binom{n-1}{k-1}^2 + \binom{n-2}{k-2}\left[\binom{n-1}{k} - \binom{n-2}{k-1}\right]\right]|E(G)| \\
= & \left[\binom{n-2}{k-1}^2 + \binom{n-2}{k-2}\binom{n-2}{k}\right]|E(G)|.
\end{aligned}$$

For $|E(G)| > 1$, take another edge $\{x',y'\} \in E(G)$. Since there are $A \in S_x$ and $B \in S_y$ such that $A \supset \{x'\}$ and $B \supset \{y'\}$, then AB is provided by $\{x,y\}$ and $\{x',y'\}$, meaning that AB is counted twice in $GF_k(G)$. Therefore,

$$|E(GF_k(G))| \leq \left[\binom{n-2}{k-1}^2 + \binom{n-2}{k-2}\binom{n-2}{k}\right]|E(G)|.$$

While the degree and size formulas established above provide a global quantitative snapshot of the graph's density and vertex connectivity, they do not fully illuminate the underlying architectural mapping. To understand how the internal geometry of G dictates the formation of $GF_k(G)$, we must shift our focus from these numerical invariants to a qualitative analysis of structural inheritance. We now examine how subgraph structures in G translate to $GF_k(G)$.

Theorem 3.8. *If H is subgraph of G, then $GF_k(H)$ is a subgraph of $GF_k(G)$.*

Proof. For any $A \in X_k(H)$, we have $A \subseteq V(H) \subseteq V(G)$, so $A \in X_k(G)$. Therefore, $X_k(H) \subseteq X_k(G)$. For any pair of adjacent vertices A, B in $GF_k(H)$. By Definition 1.1, there exists $\{x, y\} \in E(H)$ such that $\{x, y\} \subseteq A \triangle B$. Since $E(H) \subseteq E(G)$ then $\{x, y\} \in E(G)$. In other words, there exists $\{x, y\} \in E(G)$ such that $\{x, y\} \subseteq A \triangle B$. Thus, A, B is also a pair of adjacent vertices in $GF_k(G)$, meaning $E(GF_k(H)) \subseteq E(GF_k(G))$.

The following theorem identifies the presence of a vertex of degree two allows G to be embedded in $GF_2(G)$.

Theorem 3.9. *The existence of a vertex $v \in V(G)$ with $\deg_G(v) = 2$ is a sufficient condition for G to be isomorphic to a subgraph of $GF_2(G)$.*

Proof. By Theorem 2.4, the condition $\deg_G(v) = 2$ implies that G is isomorphic to a subgraph of $F_2(G)$. Furthermore, Theorem 3.2 (i) establishes that $F_2(G)$ is a spanning subgraph of $GF_2(G)$. Since the subgraph relation is transitive, it follows that G is also isomorphic to a subgraph of $GF_2(G)$.

While the previous result focuses on the preservation of the base graph G itself, the structural inheritance of $GF_k(G)$ also extends to more complex configurations involving multiple components. We now demonstrate how $GF_k(G)$ accommodates the interaction of disjoint subgraphs through the formation of Cartesian product structures. Recall that for any non-empty set C, we have $A \triangle B = (A \cap C) \triangle (B \triangle C)$.

Theorem 3.10. *Let $H_1, \ldots, H_m$ are disjoint induced subgraphs of G, then for all integers $s_1, \ldots, s_m$ such that $1 \le s_i \le |V(H_i)|$ and $\sum_{i=1}^{m} s_i = k$, it follows that $GF_{s_1}(H_1) \square \ldots \square GF_{s_m}(H_m)$ is a subgraph of $GF_k(G)$.*

Proof. Let H_1 and H_2 be two disjoint subgraphs of G induced by $W_1, W_2 \subseteq V(G)$, respectively. Let s_1 and s_2 be integers such that $1 \le s_i \le |W_i|$ and $s_1 + s_2 = k$. Let $\Gamma_i = GF_{s_1}(H_1) \square GF_{s_2}(H_2) \square \ldots \square GF_{s_i}(H_i)$ and for all $A \in X_k(G)$, $X_{s_i}(H_i) = \{A \cap W_i : |A \cap W_i| = s_i\}$. Define a function f from $V(\Gamma_2)$ to $X_k(G)$ such that for all $(A \cap W_1, A \cap W_2) \in V(\Gamma_2)$ satisfy $f(A \cap W_1, A \cap W_2) = A \cap (W_1 \cup W_2)$. Observe that f is injective. Now, take any two adjacent vertices $(A \cap W_1, A \cap W_2), (B \cap W_1, B \cap W_2) \in V(\Gamma_2)$. By definition of cartesian product of two graphs, assume that $A \cap W_1 = B \cap W_1$, so we have $A \cap W_2$ is adjacent with $B \cap W_2$ in $GF_{s_2}(H_2)$. There exists $S \in E(G)$ such that $S \subseteq (A \cap W_2) \triangle (B \cap W_2)$. Thus, $S \subseteq A \triangle B$. Therefore, $f((A \cap W_1, A \cap W_2))$ is adjacent with $f((B \cap W_1, B \cap W_2))$ in $GF_k(G)$. We have f is an injective homomorphism from $GF_{s_1}(H_1) \square GF_{s_2}(H_2)$ to $GF_k(G)$. In other words, $GF_{s_1}(H_1) \square GF_{s_2}(H_2)$ is isomorphic to a subgraph of $GF_k(G)$. By induction on m, the proof is completed.

Unlike Theorem 2.10 that $F_{s_1}(H_1) \square F_{s_m}(H_m)$ is an induced subgraph of $F_k(G)$, with the same condition of G, we have $GF_{s_1}(H_1) \square \ldots \square GF_{s_m}(H_m)$ is not always an induced subgraph of $GF_k(G)$. For example, given a graph G on 6 vertices where $V(G) = \{1, 2, 3, 4, 5, 6\}$ and $E(G) = \{\{1, 2\}, \{2, 3\}, \{4, 5\}\}$. Let H_1 and H_2 be subgraphs of G induced by $W_1 = \{1, 2, 3\}$ and $W_2 = \{4, 5\}$. We have $GF_1(H_1) \cong P_3$ and $GF_1(H_2) \cong K_2$ and then observe that maximum number of $\deg_{GF_1(H_1) \square GF_1(H_2)}(A)$ is 3, for any $A \in V(GF_1(H_1) \square GF_1(H_2))$. However, consider that for any $\{x, y\} \in GF_2(G)$, by Theorem 3.5, if $\{x, y\} \in E(G)$, then $\deg_{GF_2(G)}(\{x, y\}) \ge \binom{6-2}{2} = 6$. If $\{x, y\} \notin E(G)$, then observe that $\deg_{GF_2(G)}(\{x, y\}) > 3$. These imply that there is no induced subgraph of $GF_2(G)$ which isomorphic to $GF_1(H_1) \square GF_1(H_2)$ or equivalently, $GF_1(H_1) \square GF_1(H_2)$ is not an induced subgraph of $GF_2(G)$.

Now consider when $G = 2K_2$, let $V(G) = \{a_1, a_2, a_3, a_4\}$ and $E(G) = \{a_1, a_3\} \cup \{a_2, a_4\}$. Let H_1 and H_2 are disjoint subgraphs of G induced by $\{a_1, a_3\}$ and $\{a_2, a_4\}$, respectively, so we have $G = H_1 \cup H_2$. Observe that $GF_2(G)$ is not regular, but $GF_1(H_1) \square GF_1(H_2)$ is 2-regular. Therefore, there is no induced subgraph of $GF_2(G)$ which isomorphic to $GF_1(H_1) \square GF_1(H_2)$ or equivalently, $GF_1(H_1) \square GF_1(H_2)$ is not an induced subgraph of $GF_2(G)$. However, it is still unknown whether there exists a condition that Theorem 2.10 also hold for generalized token graphs.

The ability of $GF_k(G)$ to inherit subgraphs and product structures suggests that the generalized token graph preserves significant portions of the base graph's "DNA." However, this inheritance is particularly striking when G belongs to a well-defined family of graphs. Next, we characterize the exact topology of $GF_k(G)$ for fundamental graph classes. We derive the generalized token graphs for star graphs and complete graphs, and characterize when $GF_2(G)$ forms a tree or a cycle.

Theorem 3.11. *The following assertions are true.*

1. *If $G = K_{1,n-1}$, then $GF_k(G) = K_{\binom{n-1}{k-1}, \binom{n-1}{k}}$.*
2. *G is a complete graph if and only if $GF_k(G)$ is a complete graph.*

Proof. Consider the following.

1. Let $G = K_{1,n-1}$ be a star graph with c as its center vertex. Let $V_1 = \{\{c\} \cup A \mid A \subseteq V(G) - \{c\}, |A| = k - 1\}$ and $V_2 = \{B \subseteq V(G) - \{c\} \mid |B| = k\}$. Clearly, $|V_1| = \binom{n-1}{k-1}$ and $|V_2| = \binom{n-1}{k}$. By Pascal's formula, we obtain $|V_1| + |V_2| = \binom{n}{k} = |X_k(G)|$. Let $X, Y \in V_1$. Clearly, $X \triangle Y \subseteq V(G) \backslash \{c\}$. Since there are no edges in $E(G)$ connecting two vertices from $V(G) \backslash \{c\}$, then X and Y are not adjacent in $GF_k(G)$. Consequently, V_1 becomes an independent set. Similarly, for all $X, Y \in V_2$, we have $c \notin X \triangle Y$. Consequently, V_2 also becomes an independent set. This proves that $GF_k(G)$ is a bipartite graph with bipartition V_1 and V_2. Furthermore, for any $X \in V_1$ and $Y \in V_2$, X is adjacent to Y as there exists $x \in V(G) \backslash \{c\}$ such that $\{c, x\} \subseteq X \triangle Y$. Hence, $GF_k(G) = K_{\binom{n-1}{k-1}, \binom{n-1}{k}}$.
2. Let G be a complete graph. In other words, any pair of distinct vertices of G are adjacent. Consider for any $A, B \in X_k(G)$ where $A \ne B$, it satisfy $|A \triangle B| \ge 2$ by

Lemma 3.1. Consequently, $A \triangle B$ contains at least one pair of distinct vertices in G. Therefore, $GF_k(G)$ is also a complete graph. Conversely, let $GF_k(G)$ be a complete graph. In other words, any pair of distinct vertices of $GF_k(G)$ are adjacent. Let $x, y \in V(G)$ be any vertices with $x \neq y$. Clearly, there must exist $A = \{x, a_1, a_2, ..., a_{k-1}\}$ and $B = \{y, a_1, a_2, ..., a_{k-1}\}$ in $GF_k(G)$ such that $A \triangle B = \{x, y\}$. Since A and B are adjacent in $GF_k(G)$ then there exists $\{u, v\} \in E(G)$ such that $A \triangle B \supseteq \{u, v\}$. Consequently, $\{u, v\} = \{x, y\}$. Therefore, $\{x, y\} \in E(G)$. Since x and y are arbitrary distinct vertices in G and satisfy $\{x, y\} \in E(G)$ then G is also a complete graph.

Having characterized the generalized token graphs of star graphs and complete graphs, we now turn our attention to the case $k = 2$. In particular, we investigate conditions under which $GF_2(G)$ forms a tree or a cycle.

Theorem 3.12. *The following assertions are true.*

1. *$GF_2(G)$ is a tree if and only if $G = P_2$ or P_3.*
2. *$GF_2(G)$ is a cycle graph if and only if $G = K_3$.*

Proof. Consider the following.

1. If $GF_2(G)$ is a tree, then $F_2(G)$ is a tree. By Theorem 2.3, $G = P_2$ or P_3. Clearly if $G = P_2$ or P_3, then $GF_2(G) = P_2$ or P_3, respectively.
2. If $G = K_3$, it is clear that $GF_2(K_3) = C_3$. Let $GF_2(G)$ be a cycle graph. Suppose that $G \neq K_3$. We have the following three possibilities.
   (i). If $1 \leq n \leq 2$, then $|X_2(G)| \leq 1$. Hence $GF_2(G)$ will not be a cycle graph which contradicts the hypothesis.
   (ii). If $n = 3$ but $G \neq K_3$, then $1 \leq |E(G)| \leq 2$. Therefore, $G = P_2$ or $G = K_2 \cup N_1$. Observe that $GF_2(G)$ is not a cycle graph which contradicts the hypothesis.
   (iii). For $n > 3$, let $V(G) = \{v_i : 1 \leq i \leq n\}$. Consider for $E(G) = \{\{v_i, v_j\}\}$ for some $i, j \in \{1, 2, \ldots, \}$. If $n = 4$, then $\deg_{GF_2(G)}(\{v_i, v_j\}) = 1$. If $n > 4$, then $n - 3 \geq 2$. Thus, $\{v_1, v_i\}$ is adjacent with $\{v_1, v_j\}, \{v_j, v_l\}$ where $l \in \{1, 2, \ldots, n\} - \{1, i, j\}$. We have the number of l must be $n - 3$. Thus, $\deg_{GF_2(G)}(\{v_1, v_i\}) \geq 3$. In other words, $GF_2(G)$ is not a cycle graph for all $n > 3$, which contradicts the hypothesis. It would give the same result when $|E(G)| > 1$.

   By those possibilities, G must be equal to K_3.

Establishing that $GF_k(G)$ can result in specific topologies like complete bipartite graphs or cycle graphs leads to deeper questions regarding its global invariants. Specifically, we investigate whether higher-level properties such as the length of the shortest cycle, the existence of Hamiltonian paths, or the presence of odd cycles are maintained through this transformation. Now, we examine the global topological invariants of generalized token graphs. We specifically focus on the inheritance of cycles and cliques, the relationship between $GF_k(G)$ and classical line graphs, and the conditions for Hamiltonicity and bipartiteness.

The presence of small subgraphs in G often dictates the girth and local density of the resulting generalized token graph. We begin by showing how cycles and triangles are preserved or even induced.

Theorem 3.13. *If G contains C_4, then $GF_k(G)$ contains C_4.*

Proof. Let $V(G) = \{x_1, x_2, \ldots, x_n\}$. Let G contains C_4, say $(x_{m_1}, x_{m_2}, x_{m_3}, x_{m_4}, x_{m_1})$. For any $k \geq 2$, take any $A \subseteq V(G) - \{x_{m_1}, x_{m_2}, x_{m_3}, x_{m_4}\}$ such that we have a cycle $(\{x_{m_1}, x_{m_2}\} \cup A, \{x_{m_3}, x_{m_4}\} \cup A, \{x_{m_2}, x_{m_4}\} \cup A, \{x_{m_1}, x_{m_3}\} \cup A, \{x_{m_1}, x_{m_2}\} \cup A)$ of length 4. Therefore, $GF_k(G)$ contains C_4.

While cycles of length four are preserved, the generalized adjacency rule often forces the appearance of triangles, leading to a girth of three even when the original graph is triangle-free.

Theorem 3.14. *If $GF_k(G)$ is triangle-free, then G is also triangle-free.*

Proof. We prove the contrapositive: if G contains a triangle, then $GF_k(G)$ contains a triangle. Let $\{u, v, w\}$ be the vertices of a K_3 in G. Let $S \subseteq V(G) \setminus \{u, v, w\}$ be a set of size $k - 2$. Consider the vertices $A = \{u, v\} \cup S$, $B = \{v, w\} \cup S$, and $C = \{w, u\} \cup S$ in $X_k(G)$. We compute their symmetric differences as follows: $A \triangle B = \{u, w\}$, which is an edge in G; $B \triangle C = \{v, u\}$, which is an edge in G; $C \triangle A = \{w, v\}$, which is an edge in G. Since every pair of vertices in $\{A, B, C\}$ satisfies the adjacency condition, they form a K_3 in $GF_k(G)$. Thus, the existence of a triangle in G implies the existence of a triangle in $GF_k(G)$.

Theorem 3.15. *If G contains two independent edges and satisfies $\binom{n-4}{k-2} > 0$, then $girth(GF_k(G)) = 3$.*

Proof. Let $\{x, y\}, \{x', y'\} \in E(G)$ be two independent edges. Since $\binom{n-4}{k-2} > 0$ then there exists $H \subset X_k(G) - \{x, y, x', y'\}$ such that $|H| = k - 2$. We can find three vertices of $GF_k(G)$ which are A, B, and C such that $A = \{x, x'\} \cup H$, $B = \{x, y'\} \cup H$, and $C = \{y, y'\} \cup H$. Simply the set of those vertices induces a cycle of length 3 in $GF_k(G)$.

This local density suggests that $GF_k(G)$ is naturally inclined toward complete subgraphs. Indeed, the existence of a clique in G guarantees a corresponding clique in the generalized token graph.

Theorem 3.16. *For any $r \leq n$, if G contains K_r, then $GF_k(G)$ contains K_s, for all $1 \leq s \leq \binom{r}{k}$.*

Proof. Let G contains K_r, for any $r \leq n$. If $r < k$, then $GF_k(K_r)$ is not a graph. Therefore, we assume that $r \geq k$. From Theorem 3.11 (2), it follows that $GF_k(K_r) \cong K_{\binom{r}{k}}$. Furthermore, by Theorem 3.8, it follows that $GF_k(K_r)$ is a subgraph of $GF_k(G)$. Hence, $GF_k(G)$ contains K_s, for all $1 \leq s \leq \binom{r}{k}$.

The structural richness of $GF_k(G)$ is further highlighted by its relationship with line graphs, denoted by $L(G)$, that is a graph where $V(L(G)) = E(G)$ and $x, y \in V(L(G))$ are adjacent if and only if x and y have the same common vertex of G, in the sense that x and y have a common end vertex.

Theorem 3.17. *$GF_2(G)$ contains $L(G)$ if and only if $G = K_n$.*

Proof. As established in Theorem 2.6, the isomorphism $F_2(G) \cong L(G)$ holds if and only if G is a complete graph K_n. Observing that $F_2(G)$ is a subgraph of $GF_2(G)$, we conclude that $GF_2(G)$ contains the line graph $L(G)$ precisely when $G = K_n$.

In terms of hamiltonicity and bipartiteness of generalized token graphs, we obtain the following results.

Theorem 3.18. *$GF_k(G)$ is a complete graph if and only if $F_k(G)$ is a hamiltonian graph.*

Proof. From Theorem 3.11 (2), graph G is a complete graph if and only if $GF_k(G)$ is a complete graph. Besides that, From Theorem 2.7, we have G is a complete graph if and only if $F_k(G)$ is a Hamiltonian graph. Hence, $GF_k(G)$ is a complete graph if and only if $F_k(G)$ is a Hamiltonian graph.

Theorem 3.19. *If G contains a Hamiltonian path, n is even, and k is odd, then $GF_k(G)$ contains a Hamiltonian path.*

Proof. From Theorem 2.9, $F_k(G)$ contains a Hamiltonian path. Hence, $GF_k(G)$ contains a Hamiltonian path.

Theorem 3.20. *Let G be a connected graph. If $GF_2(G)$ is bipartite, then G is bipartite.*

Proof. If $GF_2(G)$ is bipartite, then $F_2(G)$ is bipartite. By Theorem 2.8, G is bipartite.

The converse of Theorem 3.20 is not always true as shown in the following theorem.

Theorem 3.21. *If G is a connected bipartite graph where each part has at least two vertices, then $GF_k(G)$ is not bipartite for all k.*

Proof. Let G be a connected bipartite graph with $V(G) = \{v_i : 1 \le i \le n\}$ where its parts X and Y satisfy $|X|, |Y| \ge 2$. Clearly, G contains a path of length 4 i.e. (v_1, v_2, v_3, v_4) where let say $v_1, v_3 \in X$ and $v_2, v_4 \in Y$. For all k, we can take any $R \subset V(G) - \{v_1, v_2, v_3, v_4\}$ such that there exists a cycle

$$(\{v_1, v_3\} \cup R, \{v_2, v_4\} \cup R, \{v_1, v_4\} \cup R)$$

of length 3 in $GF_k(G)$. Therefore, $GF_k(G)$ is not bipartite for all k.

4. CONCLUSIONS

In this paper, we studied the structural and topological properties of the generalized k-token graph $GF_k(G)$, which extends the classical token graph by relaxing its adjacency conditions. Starting from basic set-theoretic properties, we developed a systematic analysis leading to more refined topological invariants, thereby providing a comprehensive characterization of this class of graphs. These results establish a foundation for further investigations of $GF_k(G)$. In particular, future work may focus on the spectral properties of its adjacency matrix or on determining the chromatic number $\chi(GF_k(G))$, which would yield deeper insight into its coloring and partitioning behavior.

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