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Research Paper

Perturbed Trapezoid Like Inequalities

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Abstract
In our current research article, based on a general configuration of a 3-step Peano kernel, new versions of integral inequality of Ostrowski's type are developed for differentiable mappings that have second derivatives belong to L_∞. Then we utilized these versions to generate new perturbed trapezoid like inequalities. These new perturbed trapezoid like inequalities are proposed with error bounds smaller than and similar to those reported by previous studies. Moreover, some of the obtained perturbed trapezoid like inequalities reveal the relationship between the Euler-Maclaurin summation and the trapezoidal rule. Finally, certain implementations to numerical composite quadrature rules are provided for completeness.

Keywords
Ostrowski Inequality, Euler-Maclaurin Summation Formula, Perturbed Trapezoid Type Inequality

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1. INTRODUCTION

Establishing appropriate numerical formulae to introduce an accurate approximation of certain operators are regarded necessary to solve many mathematical problems (Albadarneh et al., 2021a,b; Batiha et al., 2022; Batiha, 2011). Definite integrals of bounded functions over given intervals are sometimes difficult or even impossible to be evaluated. In such cases, the function could be complicated or not explicitly defined. Therefore, depending on varies functional values, the quadrature rules such as trapezoidal and Simpson's rules come into play. The composite trapezoidal rule is known as follows:

$$\int_{\upsilon}^{\omega} \varphi(\kappa)\, d\kappa \cong \frac{\lambda}{2}\left[\varphi(\upsilon) + 2\sum_{i=1}^{m-1}\varphi(\zeta_i) + \varphi(\omega)\right], \tag{1}$$

where $\varphi : [\upsilon, \omega] \to \mathbb{R}$ is bounded and $\upsilon < \zeta_1 < \cdots < \zeta_{m-1} < \omega$ is an equally spaced partition of $[\upsilon, \omega]$ that generates m segments each of width $\lambda = (\omega - \upsilon)/m$.

In numerical integration theory, throughout the past few decades, Ostrowski's integral inequality (Ostrowski, 1937) and also inequalities of Ostrowski's type have been utilized to analyze errors in quadrature rules (Alshanti et al., 2022). Ostrowski's inequality can be described in the following manner.

Theorem 1 *Assume $g : [\upsilon, \omega] \to \mathbb{R}$ is a continuous mapping on $[\upsilon, \omega]$ and g' is bounded on (υ, ω), then $\forall x \in [\upsilon, \omega]$, we get*

$$\left| g(x) - \frac{1}{\omega - \upsilon}\int_{\alpha}^{\upsilon} g(x)\, dx \right| \le \left[\frac{1}{4} + \left(\frac{x - \frac{\upsilon+\omega}{2}}{\omega - \upsilon}\right)^2\right] (\omega - \upsilon)\, \|g'\|_\infty, \tag{2}$$

where

$$\|g'\|_\infty = \sup_{x\in[\upsilon,\omega]} |g'(x)|.$$

Many studies adopted techniques such as examining different abstract spaces or considering distinct Peano kernel to obtain effective bounds of Ostrowski's like inequalities as well as some perturbed well known quadrature rules. Within such sharp bounds, many of the computational works rely, basically, on the values of Lebesgue norms of the derivatives associated with the provided mappings. The idea is to estimate the error for fairly general functions by relating it to quadrature errors associated with a restricted class of functions, namely piecewise polynomials called Peano kernels. Then these kernels and the derivatives of the proposed functions can be used by the means of integration by parts to generate identities. Finally, by using the functional properties of the abstract space, in which these derivatives belong to, these identities can be converted into Ostrowski's like inequalities. Some of these studies can be found in (Alshanti and Qayyum, 2017; Alshanti et al., 2017;

Alshanti, 2018, 2019; Alshanti and Milovanovic, 2020; Alshanti, 2021; Amjad et al., 2022; Liu et al., 2016; Dragomir and Sofo, 2000; Liu and Park, 2017a; Liu and Lu, 2015; Liu and Park, 2017b; Al-Zoubi et al., 2019). Also, for more rigorous related approaches, we refer the reader to (Milovanović, 2017; Milovanović and Pečarić, 1976; Milovanović, 1975; Milovanovic, 1977; Vasić and Milovanović, 1976; Đorđević and Milovanović, 1975).

2. PREVIOUS RELATED WORKS

The inequality of Ostrowski type has been, exhaustively, studied in literatures for the cases of differentiable mappings which their second derivatives belonging to L_∞. In 2000, Dragomir and Sofo (2000) demonstrated the next inequality of Ostrowski type.

Theorem 2 *Suppose* $\Omega\colon [\nu,\omega] \to \mathbb{R}$ *be a mapping where* Ω' *is an absolutely continuous on* $[\nu,\omega]$ *and* $\Omega'' \in L_\infty[\nu,\omega]$. *Then*

$$\left|\int_\nu^\omega \Omega(s)ds - \frac{1}{2}\left(\Omega(\kappa) + \frac{\Omega(\nu)+\Omega(\omega)}{2}\right)(\omega-\nu) + \frac{\omega-\nu}{2}\left(\kappa - \frac{\nu+\omega}{2}\right)\Omega'(\kappa)\right| \le \|\Omega''\|_\infty\left(\frac{1}{3}\left|\kappa-\frac{\nu+\omega}{2}\right|^3 + \frac{(\omega-\nu)^3}{48}\right), \tag{3}$$

$\forall\kappa \in [\nu,\omega]$.

Liu and Park (2017a) introduced in the next inequality of Ostrowski's type.

Theorem 3 *Suppose* $\Omega\colon [\nu,\omega] \to \mathbb{R}$ *is a mapping where* Ω' *is an absolutely continuous on* $[\nu,\omega]$ *and* $\Omega'' \in L_\infty[\nu,\omega]$. *Then*

$$\left|\frac{1}{2}\left(\frac{\Omega(\kappa)+\Omega(\nu+\omega-\kappa)}{2} + \frac{\Omega(\nu)+\Omega(\omega)}{2}\right) - \frac{1}{2}\left(\kappa-\frac{\nu+\omega}{2}\right)\frac{\Omega'(\kappa)-\Omega'(\nu+\omega-\kappa)}{2} - \frac{1}{\omega-\nu}\int_\nu^\omega \Omega(s)ds\right| \le \left[\frac{1}{3}\frac{\left(\frac{\nu+3\omega}{4}-\kappa\right)(\kappa-\nu)^2}{(\omega-\nu)^3} + \frac{1}{3}\frac{\left(\frac{\nu+\omega}{2}-\kappa\right)^3}{(\omega-\nu)^3}\right](\omega-\nu)^2\|\Omega''\|_\infty, \tag{4}$$

$\forall \in \left[\nu, \frac{\nu+\omega}{2}\right]$.

Remark 1 *Choosing* $= \frac{\nu+\omega}{2}$ *in* (3) *and* (4) *yields*

$$\left|\frac{1}{2}\left(\Omega\left(\frac{\nu+\omega}{2}\right) + \frac{\Omega(\nu)+\Omega(\omega)}{2}\right) - \frac{1}{\omega-\nu}\int_\nu^\omega \Omega(s)\,ds\right| \le \frac{(\omega-\nu)^2}{48}\|\Omega''\|_\infty. \tag{5}$$

Remark 2 *For* $= \nu$ *in* (4), *we have the following perturbed inequality of trapezoidal type*

$$\left|\frac{\Omega(\nu)+\Omega(\omega)}{2} - \frac{(\omega-\nu)}{8}\left(\Omega'(\omega)-\Omega'(\nu)\right) - \frac{1}{\omega-\nu}\int_\nu^\omega \Omega(s)\,ds\right| \le \frac{(\omega-\nu)^2}{24}\|\Omega''\|_\infty. \tag{6}$$

The estimator of (6) is smaller than the estimator of the classical trapezoidal rule.

In this paper, motivated by Dragomir and Sofo (2000) and Liu and Park (2017a), perturbed trapezoidal type inequalities with a range of estimates are obtained through considering arbitrary parameter $\hbar \in [0,1]$. New perturbed trapezoid type inequalities are proposed with similar and smaller errors than those reported by both Dragomir and Sofo (2000) and Liu and Park (2017a), respectively.

3. MAIN RESULTS

The primary goal of this part is to establish certain integral inequalities of Ostrowski type via proposing a generalization of the 3-step linear Peano kernel (see (8) below). We consider differentiable mappings that have second derivatives belong to L_∞ to obtain the our results.

Theorem 4 *Suppose* $\phi : [\nu,\omega] \to \mathbb{R}$ *a given mapping whereby* ϕ' *is an absolutely continuous over* $[\nu,\omega]$ *and assume that* $\phi'' \in L_\infty[\nu,\omega]$. *Then* $\forall x \in \left[\nu, \nu+\hbar\frac{\omega-\nu}{2}\right]$ *and* $\hbar \in [0,1]$, *we have:*

$$\Bigg|\frac{1}{4}\Big[\phi(x)+\phi(\nu)+\phi(\omega)+\phi(\nu+\omega-x)+2\hbar(\phi(\omega)-\phi(x))\Big] - \frac{1}{4}\Bigg[(1-2\hbar)\left(x-\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right)\right)\phi'(x) - \left(x-\left(\frac{\nu+\omega}{2}+\hbar\frac{\omega-\nu}{2}\right)\right)\phi'(\nu+\omega-x) + \frac{(\omega-\nu)}{2}\hbar\Big[(\hbar-1)\phi'(\nu)+(\hbar+1)\phi'(\omega)\Big]\Bigg] - \frac{1}{\omega-\nu}\int_\nu^\omega \phi(t)dt\Bigg| \le \Bigg[\frac{2}{3}\left(\frac{\nu+\omega}{2}-x\right)^3 + \frac{1}{3}\left(\frac{3\nu+\omega}{4}+\hbar\frac{\omega-\nu}{2}-x\right)^3 + \frac{1}{3}\left(\frac{3\nu+\omega}{4}-x\right)^3 + \frac{(\omega-\nu)^2}{16}\left((2\hbar-1)^2+1\right)\left(x-\left(\nu+\hbar\frac{\omega-\nu}{2}\right)\right) + \frac{(\omega-\nu)^3}{192}\left(12\hbar^2+12\hbar-2\right)\Bigg]\frac{\|\phi''\|_\infty}{2(\omega-\nu)}. \tag{7}$$

Proof: Herein, we outline the kernel $P(x,t) : [\nu,\omega] \to \mathbb{R}$ by

$$P(x,t) = \begin{cases} t-\left(\nu+\hbar\frac{\omega-\nu}{2}\right), & t\in[\nu,x], \\ t-\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right), & t\in(x,\nu+\omega-x], \\ t-\left(\omega-\hbar\frac{\omega-\nu}{2}\right), & t\in(\nu+\omega-x,\omega], \end{cases} \tag{8}$$

$\forall x \in \left[\nu, \nu + \hbar\frac{\omega-\nu}{2}\right]$ and $\hbar \in [0,1]$. Then we have (Alshanti and Qayyum, 2017):

$$\frac{1}{\omega-\nu}\int_{\nu}^{\omega} P(x,t)g'(t)\,dt = \frac{1}{2}\left[(1-2\hbar)\,g(x) + g(\nu+\omega-x) + \hbar(g(\nu) + g(\omega))\right] - \frac{1}{\omega-\nu}\int_{\nu}^{\omega} g(t)\,dt. \tag{9}$$

On choosing

$$g(x) = \left(x - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right)\phi'(x)$$

in equality (9), we get

$$\begin{aligned}
&\frac{1}{\omega-\nu}\int_{\nu}^{\omega} P(x,t)\left[\phi'(t) + \left(t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right)\phi''(t)\right]dt \\
&= \frac{1}{2}\left[(1-2\hbar)\left(x - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right)\phi'(x)\right. \\
&+ \left(\left(\frac{\nu+\omega}{2} + \hbar\frac{\omega-\nu}{2}\right) - x\right)\phi'(\nu+\omega-x) \\
&\left.+\frac{\omega-\nu}{2}\left(\left(\hbar^2-\hbar\right)\phi'(\nu) + \left(\hbar^2+\hbar\right)\phi'(\omega)\right)\right] \\
&- \frac{1}{2}\left[(1+\hbar)\,\phi(\omega) - (\hbar-1)\,\phi(\nu)\right] + \frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)dt.
\end{aligned} \tag{10}$$

Also, by considering (9), we obtain

$$\begin{aligned}
&\frac{1}{\omega-\nu}\int_{\nu}^{\omega} P(x,t)\left[\phi'(t) + \left(t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right)\phi''(t)\right]dt \\
&= \frac{1}{\omega-\nu}\int_{\nu}^{\omega} P(x,t)\phi'(t)\,dt + \frac{1}{\omega-\nu}\int_{\nu}^{\omega} P(x,t)\Big(t- \\
&\left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\Big)\phi''(t)\,dt \\
&= \frac{1}{2}\left[(1-2\hbar)\,\phi(x) + \phi(\nu+\omega-x) + \hbar\left(\phi(\nu) + \phi(\omega)\right)\right] \\
&\quad - \frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)dt + \frac{1}{\omega-\nu}\int_{\nu}^{\omega} P(x,t)\Big(t- \\
&\left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\Big)\phi''(t)\,dt.
\end{aligned} \tag{11}$$

Hence, from (10) and (11), we can get

$$\begin{aligned}
&\frac{1}{2(\omega-\nu)}\int_{\nu}^{\omega} P(x,t)\left(t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right)\phi''(t)\,dt \\
&= \frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)dt - \frac{1}{2}\Big[\frac{(1-2\hbar)\,\phi(x) + \phi(\nu+\omega-x)}{2} + \\
&\frac{\phi(\nu) + (2\hbar+1)\,\phi(\omega)}{2}\Big] + \frac{1}{2}\Big[\frac{1-2\hbar}{2}\Big(x - \Big(\frac{\nu+\omega}{2} - \\
&\hbar\frac{\omega-\nu}{2}\Big)\Big)\phi'(x) + \frac{1}{2}\left(\left(\frac{\nu+\omega}{2} + \hbar\frac{\omega-\nu}{2}\right) - x\right) \\
&\times\phi'(\nu+\omega-x) + \hbar\frac{\omega-\nu}{2}\frac{(\hbar-1)\,\phi'(\nu) + (1+\hbar)\,\phi'(\omega)}{2}\Big].
\end{aligned} \tag{12}$$

Now, applying Hölder inequality yields

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\frac{(1-2\hbar)\,\phi(x) + \phi(\nu+\omega-x)}{2} + \frac{\phi(\nu) + (2\hbar+1)\,\phi(\omega)}{2}\right] \\
&- \frac{1}{2}\Big[\frac{1-2\hbar}{2}\left(x - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right)\phi'(x) \\
&+ \frac{1}{2}\left(\left(\frac{\nu+\omega}{2} + \hbar\frac{\omega-\nu}{2}\right) - x\right)\phi'(\nu+\omega-x) \\
&+ \frac{\omega-\nu}{2}\frac{\left((\hbar^2-\hbar)\,\phi'(\nu) + (\hbar^2+\hbar)\,\phi'(\omega)\right)}{2}\Big] \\
&- \frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)dt\Bigg| \le \frac{\|\phi''\|_{\infty}}{2(\omega-\nu)}\int_{\nu}^{\omega}|P(x,t)| \\
&\left|t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right|dt.
\end{aligned} \tag{13}$$

But

$$\begin{aligned}
&\int_{\nu}^{\omega}|P(x,t)|\left|t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right|dt \\
&= \int_{\nu}^{x}\left|t - \left(\nu + \hbar\frac{\omega-\nu}{2}\right)\right|\left|t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right|dt \\
&+ \int_{x}^{\nu+\omega-x}\left[t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right]^2 dt \\
&+ \int_{\nu+\omega-x}^{\omega}\left|\left(t - \left(\omega - \hbar\frac{\omega-\nu}{2}\right)\right)\right|\left|t - \left(\frac{\nu+\omega}{2} - \hbar\frac{\omega-\nu}{2}\right)\right|dt.
\end{aligned} \tag{14}$$

Now, as $x \in \left[\nu, \nu + \hbar\frac{\omega-\nu}{2}\right]$, we can obtain

$$\begin{aligned}&\int_{\nu}^{x}\left|t-\left(\nu+\hbar\frac{\omega-\nu}{2}\right)\right|\left|t-\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right)\right|dt\\&=\int_{\nu}^{\nu+\hbar\frac{\omega-\nu}{2}}\left[\left(\nu+\hbar\frac{\omega-\nu}{2}\right)-t\right]\left[\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right)-t\right]dt\\&+\int_{\nu+\hbar\frac{\omega-\nu}{2}}^{x}\left[t-\left(\nu+\hbar\frac{\omega-\nu}{2}\right)\right]\left[\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right)-t\right]dt,\end{aligned} \tag{15}$$

and

$$\begin{aligned}&\int_{\nu+\omega-x}^{\omega}\left|\left(t-\left(\omega-\hbar\frac{\omega-\nu}{2}\right)\right)\right|\left|t-\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right)\right|dt\\&=\int_{\nu+\omega-x}^{\omega-\hbar\frac{\omega-\nu}{2}}\left[\left(\omega-\hbar\frac{\omega-\nu}{2}\right)-t\right]\left[t-\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right)\right]dt\\&+\int_{\omega-\hbar\frac{\omega-\nu}{2}}^{\omega}\left[\left(t-\left(\omega-\hbar\frac{\omega-\nu}{2}\right)\right)\right]\left[t-\left(\frac{\nu+\omega}{2}-\hbar\frac{\omega-\nu}{2}\right)\right]dt.\end{aligned} \tag{16}$$

Therefore, by referring to (13)–(16), we obtain the result (7).

4. SOME NEW PERTURBED TRAPEZOID INEQUALITIES

In this part, some novel perturbed trapezoid type inequalities are provided in light of the theoretical aspects discussed in the previous section.

Corollary 1 *Using the same assumptions of Theorem 4 and setting*

(1) $\hbar = 0$ and $x = \nu$, we get

$$\begin{aligned}&\left|\frac{\phi(\nu)+\phi(\omega)}{2}-\frac{(\omega-\nu)}{8}\left(\phi'(\omega)-\phi'(\nu)\right)\right.\\&\left.-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\le\frac{(\omega-\nu)^2}{24}\|\phi''\|_\infty\end{aligned} \tag{17}$$

which is similar to (6) that is obtained in Liu and Park (2017a). We should point out that (17) has a smaller estimator than the classical trapezoidal rule stated in Liu and Lu (2015), i.e.

$$\left|\frac{\phi(\nu)+\phi(\omega)}{2}-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\le\frac{(\omega-\nu)^2}{8}\|\phi''\|_\infty .$$

(2) $\hbar = 0$ and $x = \frac{\nu+\omega}{2}$, we have

$$\begin{aligned}&\left|\frac{1}{2}\left[\phi\left(\frac{\nu+\omega}{2}\right)+\frac{\phi(\nu)+\phi(\omega)}{2}\right]-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\\&\le\frac{(\omega-\nu)^2}{48}\|\phi''\|_\infty ,\end{aligned} \tag{18}$$

which is similar to result that is obtained in Dragomir and Sofo (2000).

(3) $\hbar = \frac{1}{2}$ and $x = \nu$, we have

$$\begin{aligned}&\left|\frac{\phi(\nu)+3\phi(\omega)}{4}-\frac{(\omega-\nu)}{32}\left(9\phi'(\omega)-\phi'(\nu)\right)\right.\\&\left.-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\le\frac{29(\omega-\nu)^2}{384}\|\phi''\|_\infty .\end{aligned} \tag{19}$$

(4) $\hbar = \frac{1}{2}$ and $x = \frac{3\nu+\omega}{4}$, we have

$$\begin{aligned}&\left|\frac{1}{4}\left[\phi(\nu)+\phi\left(\frac{\nu+3\omega}{4}\right)+2\phi(\omega)\right]-\frac{(\omega-\nu)}{32}\left[4\phi'\left(\frac{\nu+3\omega}{4}\right)\right.\right.\\&\left.\left.+3\phi'(\omega)-\phi'(\nu)\right]-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\\&\le\frac{5(\omega-\nu)^2}{192}\|\phi''\|_\infty .\end{aligned} \tag{20}$$

(5) $\hbar = 1$ and $x = \nu$, we have

$$\begin{aligned}&\left|\frac{\phi(\omega)}{4}-\frac{(\omega-\nu)}{4}\left(2\phi'(\nu)+\phi'(\omega)\right)-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\\&\le\frac{35(\omega-\nu)^2}{384}\|\phi''\|_\infty .\end{aligned} \tag{21}$$

(6) $\hbar = 1$ and $x = \frac{\nu+\omega}{2}$, we have

$$\begin{aligned}&\left|\frac{\phi(\nu)+3\phi(\omega)}{4}-\frac{(\omega-\nu)}{4}\phi'(\omega)-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\\&\le\frac{11(\omega-\nu)^2}{192}\|\phi''\|_\infty .\end{aligned} \tag{22}$$

Remark 3 *In (19), if we consider $\phi'(\nu) = \phi'(\omega)$ and utilize the three-point midpoint approximation of $\phi'\left(\frac{\nu+2\omega}{4}\right)$, then we generate the next perturbed inequality of trapezoid type:*

$$\begin{aligned}&\left|\left[\frac{\phi(\nu)+\phi(\omega)}{2}+\frac{\phi\left(\frac{\nu+\omega}{2}\right)+\phi\left(\frac{\nu+3\omega}{4}\right)}{2}\right]-\frac{1}{\omega-\nu}\int_{\nu}^{\omega}\phi(t)\,dt\right|\\&\le\frac{5(\omega-\nu)^2}{192}\|\phi''\|_\infty .\end{aligned} \tag{23}$$

Remark 4 *Inequality (23) represents a new perturbed trapezoid type inequality with right hand side has smaller estimator than that of (6).*

Remark 5 *Inequality (17) reveals the relationship between the Euler-Maclaurin summation formula and the trapezoidal rule as*

$$\sum_{x=\nu}^{\omega}\phi(x) = \int_{\nu}^{\omega}\phi(t)\,dt + \frac{1}{2}\left[\phi(\nu)+\phi(b)\right] + \sum_{i=2}^{k}\frac{b_i}{i!}\left[\phi^{(i-1)}(\omega)-\phi^{(i-1)}(\nu)\right] - \int_{\nu}^{\omega}\frac{B_k(\{1-t\})}{k!}\phi^{(k)}(t)\,dt, \tag{24}$$

where B_n represent the Bernoulli polynomials, b_n represent some given numbers, $\nu,\omega\in\mathbb{R}$ such that $\omega-\nu\in\mathbb{Z}^+$, $k\in\mathbb{Z}^+$, and the symbol $\{\zeta\}$ indicates to the fractional term of ζ in which $\zeta\in\mathbb{R}$ (Milovanovi, 2017).

5. APPLICATIONS TO COMPOSITE QUADRATURE RULES

In this following content, we utilize our new result (7) to carry out some novel composite quadrature formulae with much lower error.

Theorem 5 *Assume $\varphi_n : \nu = \upsilon_0 < \upsilon_1 < \upsilon_2 < \cdots < \upsilon_{n-1} < \upsilon_n = \omega$ is a partition of $[\nu,\omega]$, $\Delta_i = \upsilon_{i+1}-\upsilon_i$ $(i = 0, 1, \cdots, n-1)$, $\eta(\Delta) := \max\{\Delta_i : i = 0, 1, \cdots, n\}$, $\xi_i \in [\upsilon_i, \frac{\upsilon_i+\upsilon_{i+1}}{2}]$, $\hbar\in[0,1]$ and*

$$\begin{aligned}\Psi(\phi,\varphi_n,\xi) =& \frac{1}{4}\sum_{i=0}^{n-1}\Delta_i\big[\phi(\xi_i)+\phi(\upsilon_i)+\phi(\upsilon_{i+1}) + \phi(\upsilon_i+\upsilon_{i+1}-\xi_i)+2\hbar\left(\phi(\upsilon_{i+1})-\phi(\xi_i)\right)\big]\\ &-\frac{1}{4}\sum_{i=0}^{n-1}\Delta_i\Bigg[(1-2\hbar)\left(\xi_i-\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}-\frac{\hbar}{2}\Delta_i\right)\right)\phi'(\xi_i)-\left(\xi_i-\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}+\frac{\hbar}{2}\Delta_i\right)\right)\\ &\phi'(\upsilon_i+\upsilon_{i+1}-\xi_i)+\frac{\Delta_i}{2}\left(\left(\hbar^2-\hbar\right)\phi'(\upsilon_i) + \left(h^2+\hbar\right)\phi'(\upsilon_{i+1})\right)\Bigg],\end{aligned} \tag{25}$$

then

$$\int_{\nu}^{\omega}\phi(\upsilon)\,d\upsilon = \Psi(\phi,\varphi_n,\xi)+\$(\phi,\varphi_n,\xi), \tag{26}$$

where $\$(\phi,\varphi_n,\xi)$ represents the remainder satisfying the following estimate

$$\begin{aligned}\$(\phi,\varphi_n,\xi) \le& \frac{\|\phi''\|_\infty}{2}\Bigg\{\frac{1}{3}\sum_{i=0}^{n-1}\Bigg[2\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}-\xi_i\right)^3 + \left(\frac{3\upsilon_i+\upsilon_{i+1}}{4}+\hbar\Delta_i-\xi_i\right)^3+\left(\frac{3\upsilon_i+\upsilon_{i+1}}{4}-\xi_i\right)^3\Bigg]\\ &+\frac{1}{16}\sum_{i=0}^{n-1}\Delta_i^2\Big[\left((2\hbar-1)^2+1\right)(\xi_i-(\upsilon_i+\hbar\Delta_i)) + \frac{\Delta_i}{12}\left(12\hbar^2+12\hbar-2\right)\Big]\Bigg\}.\end{aligned} \tag{27}$$

Proof: By applying (7) on $\xi_i\in[\upsilon_i, \frac{\upsilon_i+\upsilon_{i+1}}{2}]$, we obtain

$$\begin{aligned}&\Bigg|\int_{\upsilon_i}^{\upsilon_{i+1}}\phi(t)dt-\frac{\Delta_i}{4}\Big[\phi(\xi_i)+\phi(\upsilon_i)+\phi(\upsilon_{i+1}) + \phi(\upsilon_i+\upsilon_{i+1}-\xi_i)\,2\hbar\left(\phi(\upsilon_{i+1})-\phi(\xi_i)\right)\Big]\\ &+\frac{\Delta_i}{4}\Bigg[(1-2\hbar)\left(\xi_i-\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}-\frac{\hbar}{2}\Delta_i\right)\right)\phi'(\xi_i) - \left(\xi_i-\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}+\frac{\hbar}{2}\Delta_i\right)\right)\phi'(\upsilon_i+\upsilon_{i+1}-\xi_i)\\ &+\frac{\Delta_i}{2}\left(\left(\hbar^2-\hbar\right)\phi'(\upsilon_i)+\left(h^2+\hbar\right)\phi'(\upsilon_{i+1})\right)\Bigg]\Bigg| \le\\ &\Bigg\{\frac{1}{3}\Bigg[2\left(\frac{\upsilon_i+\upsilon_{i+1}}{2}-\xi_i\right)^3+\left(\frac{3\upsilon_i+\upsilon_{i+1}}{4}+\hbar\Delta_i-\xi_i\right)^3 + \left(\frac{3\upsilon_i+\upsilon_{i+1}}{4}-\xi_i\right)^3\Bigg]\\ &+\frac{\Delta_i^2}{16}\Bigg[\left((2\hbar-1)^2+1\right)(\xi_i-(\upsilon_i+\hbar\Delta_i)) + \frac{\Delta_i}{12}\left(12\hbar^2+12\hbar-2\right)\Bigg]\Bigg\}\frac{\|\phi''\|_\infty}{2}.\end{aligned} \tag{28}$$

$\forall i = 0, 1, \cdots, n-1$. Consequently, by taking the summation from 0 to $n-1$ over i coupled with using the triangle inequality, then (26) can be obtained.

Corollary 2 *Using the same assumptions of Theorem 7 and selecting $\xi_i=\upsilon_i$ with $\hbar=\frac{1}{2}$, we have*

$$\Psi(\phi,\varphi_n,\xi) = \frac{1}{4}\sum_{i=0}^{n-1}\Big(\left[\phi(\upsilon_i)+3\phi(\upsilon_{i+1})\right]-\frac{\Delta_i}{8}\left(9\phi'(\upsilon_{i+1}) - \phi'(\upsilon_i)\right)\Big) \tag{29}$$

and

$$\$ (\phi, \varphi_n) \leq \frac{29 \|\phi''\|_\infty}{384} \sum_{i=0}^{n-1} \Delta_i^3. \quad (30)$$

6. CONCLUSION

Motivated by Dragomir and Sofo (2000) and Liu and Park (2017a), new Ostrowski type integral inequality is obtained for differentiable mappings which their second derivatives belong to L_∞. Our result reveals a range of estimates along with what it was provided in Alshanti (2018) and Alshanti (2019). We established new perturbed trapezoid like inequalities through utilizing a parameter $\hbar \in [0, 1]$. The new perturbed trapezoid type inequalities are proposed with error bounds smaller than and similar to those reported by previous studies, namely, Dragomir and Sofo (2000) and Liu and Park (2017a), respectively. Certain implementations to composite quadrature rules are carried out as well. Our future work is to obtain some better estimates of other well known quadrature rules such as the Simpson's rules.

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