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Research Paper

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A New Three-Parameter Mixed Poisson Transmuted Weighted Exponential Distribution with Applications to Insurance Data

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Abstract
The classical Poisson distribution assumes equidispersion and this is hardly the case when given count observations. To improve efficiency and general applicability for dispersed data, this paper obtains a new mixing probability distribution using quadratic transmutation map on the weighed exponential distribution. The mixing distribution is used to obtain a new three parameter mixed Poisson distribution. The new distribution has unimodal shape with positive skewness and tends to zero speedily. Basic properties of the new distribution like the distribution function, moments, and dispersion index are obtained. Maximum likelihood estimates of the parameters of the distribution are obtained. Applications are made using four claims frequencies from different insurance companies. Results show that the new distribution gives a good fit in comparisons with some referred discrete distributions.

Keywords
Weighted Exponential Distribution, Quadratic Transmutation Map, Dispersed Count Observations, Parameter Estimation

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1. INTRODUCTION

A major aim in exploring probability distribution is to obtain new distribution that gives higher efficiency to data being modelled. The objective is to obtain a new distribution that gives more flexibility and general applicability to real life observations. To this end, several techniques on improving classical distributions pervade literature. Among several methods of compounding classical probability distributions are: the quadratic transmutation map (Shaw and Buckley, 2007); the beta extended G distributions (Cordeiro et al., 2012); the beta exponential G distributions (Alzaatreh et al., 2013); the exponentiated generalized G distributions (Cordeiro et al., 2013); the Weibull-G family (Bourguignon et al., 2014); the transmuted exponentiated generalized-G (Yousof et al., 2015); the alpha-power Transformation (Mahdavi and Kundu, 2017); and the cubic rank transmutations (AL Kadim, 2018; Aslam et al., 2018; Granzotto et al., 2017; Rahman et al., 2019).

In count observation paradigm, the Poisson distribution has received wide acceptance largely due to its simple form and ease of parameter interpretation. The distribution however assumes equality for mean and variance which is rarely the case when given real count observations. To improve this, several methods have been proposed to modify the Poisson distribution. The aim is usually to obtain a more flexible distribution that can model dispersed observation. Popular among these is the mixed Poisson distribution(Greenwood and Yule, 1920). The mixed Poisson always have higher variance than mean (Karlis and Xekalaki, 2005), hence, would be efficient to model over-dispersed observations which characterize several count distributions, especially claims frequency in actuary science (Adcock et al., 2015; Adetunji and Sabri, 2021; Omari et al., 2018). The process involves assuming a distribution (the mixing distribution) with positive supports for the parameter of the Poisson distribution. Conditional distribution of the joint distribution of the Poisson distribution and the mixing distribution is then obtained. Different forms of the mixed Poisson distribution have been proposed with evidences of providing more efficiency in modelling dispersed count data (Bhati et al., 2015; Karlis and Xekalaki, 2005; Zakerzadeh and Dolati, 2009).

This research uses the Quadratic Transmutation (QT) map of Shaw and Buckley (2007) on the weighted exponential distribution Gupta and Kundu (2009) to obtain a new mixing probability distribution. The new distribution is then assumed for the parameter of the Poisson distribution. This is then used to obtain a new three-parameter mixed Poisson distribution.

2. EXPERIMENTAL SECTION

2.1 Methodology

2.1.1 Weighted Exponential Distribution

Using the process proposed by Azzalini (1985), the Weighted Exponential Distribution (WED) was proposed by Gupta and Kundu (2009). The shape of the distribution is similar to other common two-parameter lifetime distributions like the Weibull and Gamma distributions. The distribution has been reported to perform favourably well like other lifetime distributions (Altun, 2019; Gupta and Kundu, 2009; Shakhatreh, 2012). If a continuous random variable Y assumes the WED, its distribution function is defined in Equation (1) as:

$$G(x) = 1 - \frac{1}{\lambda} e^{-\theta}(\lambda + 1 - e^{-\lambda\theta y}) \tag{1}$$

2.1.2 Quadratic Transmutation Map

The Quadratic Transmutation (QT) map was proposed by Shaw and Buckley (2007). If a baseline distribution has a distribution function denoted with G(y), the equivalent QT is obtained using Equation (2):

$$F(y) = (1+\alpha)G(y) - \alpha(G(y))^2; y \in \mathbb{R}|\alpha| \leq 1 \tag{2}$$

2.1.3 Quadratic Transmuted Weighted Exponential Distribution

Given the WED as in (1), the Quadratic Transmuted Weighted Exponential Distribution (QTWED) is obtained by inserting (1) into (2). Hence, the distribution function for a random variable Y that assumes the QTWED is presented in Equation (3) while its PDF is presented in Equation (4) as:

$$F(y) = 1 - \frac{\alpha(1+\alpha)^2}{\lambda^2} e^{-\theta y} - \frac{\alpha}{\lambda^2} e^{-2(1+\lambda)\theta y} - \frac{(\alpha-1)}{\lambda} e^{-(1+\lambda)\theta y} + \frac{2\alpha(1+\lambda)}{\lambda^2} e^{-(2+\lambda)\theta y} \tag{3}$$

$$F(y) = \frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2} e^{-(1+\lambda)\theta y} - \frac{\lambda(\alpha-1)}{2} e^{-\theta y} + \alpha e^{-2(1+\lambda)\theta y} + \alpha(1+\lambda)e^{-2\theta y} - \alpha(2+\lambda)e^{-(2+\lambda)\theta y}\right) \tag{4}$$

The survival and hazard functions are respectively given in Equations (5) and (6) as:

$$S(y) = \frac{(\lambda(\alpha-1)e^{-(1+\lambda)\theta y} 2\alpha(1+\lambda)e^{-(2+\lambda)\theta y} +}{\lambda^2}$$

$$\frac{\alpha e^{-2(1+\lambda)\theta y} + (1+\lambda)(\alpha(1+\lambda)e^{-\theta y} - \lambda(\alpha-1)e^{-\theta y})}{\lambda^2} \tag{5}$$

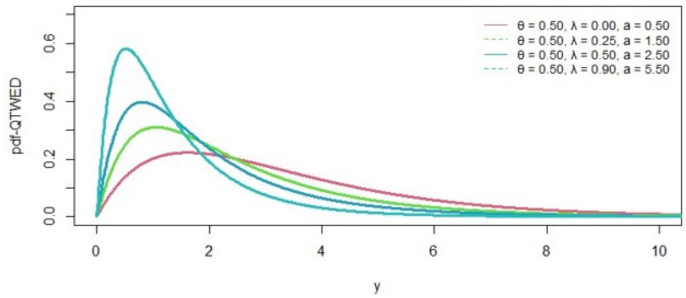

Figure 1. Shapes of the PDF of the QTWED

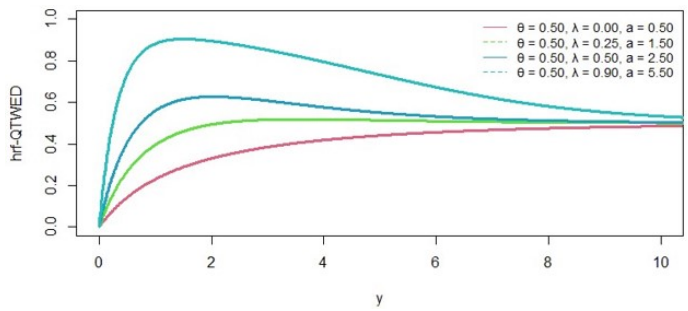

Figure 2. Shapes of the HRF of the QTWED

$$h(y) = \frac{2\theta(1+\lambda)\left(\frac{\lambda(\alpha-1)}{2} e^{-(1+\lambda)\theta y} - \lambda\frac{(\alpha-1)}{2} e^{-\theta y} + \alpha\right.}{(\lambda(\alpha-1)e^{-(1+\lambda)\theta y} - 2\alpha(1+\lambda)e^{-(2+\lambda)\theta y} +)}$$

$$\frac{e^{-2(1+\lambda)\theta y} + \alpha e^{-2(1+\lambda)\theta y} + \alpha(1+\lambda)e^{-2\theta y}}{\alpha e^{-2(1+\lambda)\theta y} + (1+\lambda)(\alpha(1+\lambda)e^{-2\theta y}}$$

$$\frac{\left.-\alpha(2+\lambda)e^{(2+\lambda)\theta y}\right)}{-\lambda(\alpha-1)e^{-\theta y}))} \tag{6}$$

Shapes of the PDF of the QTWED for different parameters combinations as shown in Figure 1 are positively skewed and unimodal. Figure 2 shows that the hazard rate function of the QTWED has an inverted J-shape.

2.1.4 Moment and Moment Generating Function

Proposition 1: If a random variable Y has a QTWED, the r^{th} moment is given in Equation (7) as:

$$E(Y^r) = \frac{(1+\lambda)r!}{\lambda^2\theta^r}\left(\frac{\lambda(\alpha-1)}{(1+\lambda)^{r+1}} - \lambda(\alpha-1) + \frac{\alpha}{2^2(1+\lambda)^{r+1}} + \frac{\alpha(1+\lambda)}{2^r} - \frac{2\alpha}{(2+\lambda)^r}\right) \tag{7}$$

Proof:

$$E(Y^r) = \int_0^\infty y^r f(y)dy$$

$$= \frac{2\theta(1+\lambda)}{\lambda^2}\int_0^\infty y^r\left(\frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2}e^{-(1+\lambda)\theta y}- \frac{\lambda(\alpha-1)}{2}e^{\theta y}+\alpha e^{-2(1+\lambda)\theta y}+\alpha(1+\lambda)e^{-2\theta y}-\alpha (2+\lambda)e^{-(2+\lambda)\theta y}\right)\right)$$

$$= \frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2}\int_0^\infty y^r e^{-1(1+\lambda)\theta y}dy- \frac{\lambda(\alpha-1)}{2}\int_0^\infty y^r e^{-\theta y}dy+\alpha\int_0^\infty y^r e^{-2(1+\lambda)\theta y}dy+ \alpha(1+\lambda)\int_0^\infty y^r e^{-2\theta y}dy-\alpha(2+\lambda)\int_0^\infty y^r e^{-(2+\lambda)\theta y}dy\right)$$

$$= \frac{2\theta(1+\lambda)r!}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2(1+\lambda)^{r+1}\theta^{r+1}}-\frac{\lambda(\alpha-1)}{2\theta^{r+1}}+ \frac{\alpha}{2^{r+1}(1+\lambda)^{r+1}\theta^{r+1}}+\frac{\alpha(1+\lambda)}{2^{r+1}\theta^{r+1}}-\frac{\alpha(2+\lambda)}{(2+\lambda)^{r+1}\theta^{r+1}}\right)$$

$$= \frac{2\theta(1+\lambda)r!}{\lambda^2\theta^r}\left(\frac{\lambda(\alpha-1)}{2(1+\lambda)^{r+1}}-\frac{\lambda(\alpha-1)}{2}+ \frac{\alpha}{2^{r+1}(1+\lambda)^{r+1}}+\frac{\alpha(1+\lambda)}{2^{r+1}}-\frac{\alpha}{(2+\lambda)^{r+1}}\right)$$

$$= \frac{\theta(1+\lambda)r!}{\lambda^2\theta^r}\left(\frac{\lambda(\alpha-1)}{2(1+\lambda)^{r+1}}-\lambda(\alpha-1)+ \frac{\alpha}{2^r(1+\lambda)^{r+1}}+\frac{\alpha(1+\lambda)}{2^r}-\frac{2\alpha}{(2+\lambda)^{r+1}}\right)$$

Proposition 2: If a random variable Y has a QTWED, the MGF is obtained in Equation (8) as:

$$E(e^{ty}) = \frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2(\theta+\lambda\theta-t)}-\frac{\lambda(\alpha-1)}{2(\theta-t)} +\frac{\alpha}{(2\theta+2\lambda\theta-t)}+\frac{\alpha(1+\lambda)}{(2\theta-t)}-\frac{\alpha(2+\lambda)}{(2\theta+\lambda\theta-t)}\right) \tag{8}$$

Proof:

$$E(e^{ty}) = \int_0^\infty e^{ty}f(y)dy$$

$$= \frac{2\theta(1+\lambda)}{\lambda^2}\int_0^\infty e^{ty}\left(\frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2}e^{-(1+\lambda)\theta y} -\frac{\lambda(\alpha-1)}{2}e^{\theta y}+\alpha e^{-2(1+\lambda)\theta y}+\alpha(1+\lambda)e^{-2\theta y} -\alpha(2+\lambda)e^{-(2+\lambda)\theta y}\right)\right)dy$$

$$= \frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2}\int_0^\infty e^{-(\theta+\lambda\theta-t)y}dy-\frac{\lambda(\alpha-1)}{2} \int_0^\infty e^{-(\theta-t)y}dy+\alpha\int_0^\infty e^{-(2\theta+2\lambda\theta-t)y}dy+\alpha(1+\lambda)\int_0^\infty e^{-(2\theta-t)y}dy-\alpha(2+\lambda)\int_0^\infty e^{-(2\theta+\lambda\theta-t)y}dy\right)$$

$$= \frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2(\theta+\lambda\theta-t)}-\frac{\lambda(\alpha-1)}{2(\theta-t)}+ \frac{\alpha}{2\theta+2\lambda\theta-t}+\frac{\alpha(1+\lambda)}{(2\theta-t)}-\frac{\alpha(2+\lambda)}{(2\theta+\lambda\theta-t)}\right)$$

Hence, the mean and variance of QTWED are respectively given in Equations (9) and (10) as:

$$E(Y) = \frac{(2-\alpha)\lambda^2+(8-3\alpha)\lambda+8-3\alpha}{2\theta(1+\lambda)(2+\lambda)} \tag{9}$$

$$Var(Y) = \frac{32-12\alpha-9\alpha^2+(64-24\alpha-18\alpha^2)\lambda+}{2\theta(1+\lambda)(2+\lambda)^2}$$

$$Var(Y) = \frac{(56-24\alpha-15\alpha^2)\lambda^2+(24-12\alpha-6\alpha^2)}{2\theta(1+\lambda)(2+\lambda)^2}$$

$$Var(Y) = \frac{\lambda^3+(4-2\alpha-\alpha^2)\lambda^4}{2\theta(1+\lambda)(2+\lambda)^2} \tag{10}$$

2.2 Mixed Poisson QTWED

Proposition 3: If random variable X has a Poisson (Y) distribution, and Y QTWED(θ,λ,α), the PMF of a discrete random variable X with the mixed Poisson-QTWED is given in Equation (11) as:

$$t(x) = \frac{\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{(1+\theta+\lambda\theta)^{x+1}}-\frac{\lambda(\alpha-1)}{(1+\theta)^{x+1}}+ \frac{2\alpha}{(1+2\theta+2\lambda\theta)^{x+1}}+\frac{2\alpha(1+\lambda)}{(1+2\theta)^{x+1}}-\frac{2\alpha(2+\lambda)}{(1+2\theta\theta\lambda)^{x+1}}\right) \tag{11}$$

Proof

$$t(x) = \int_0^\infty t(x|y)\times f(y)dy$$

$$t(x) = \int_0^\infty \frac{y^x e^{-y}}{x!}\left(\frac{2\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{2}e^{-(1+\lambda)\theta y} -\frac{\lambda(\alpha-1)}{2}e^{\theta y}+\alpha e^{-2(1+\lambda)\theta y}+\alpha(1+\lambda)e^{-2\theta y} -\alpha(2+\lambda)e^{-(2+\lambda)\theta y}\right)\right)dy$$

$$= \frac{2\theta(1+\lambda)}{\lambda^2 x!} \int_0^\infty \left(\frac{\lambda(\alpha-1)}{2} y^x e^{-(1+\theta+\lambda\theta)y} - \frac{\lambda(\alpha-1)}{2} \right.$$
$$y^x e^{-(1+\theta)y} + \alpha y^x e^{-(1+2\theta+2\lambda\theta)y} + \alpha(1+\lambda)$$
$$\left. y^x e^{-(1+2\theta)y} - \alpha(2+\lambda) y^x e^{-(1+2\theta+\lambda\theta)y} \right) dy$$

$$= \frac{2\theta(1+\lambda)}{\lambda^2 x!} \left(\frac{\lambda(\alpha-1)x!}{2(1+\theta+\lambda\theta)^{x+1}} - \frac{\lambda(\alpha-1)x!}{2(1+\theta)^{x+1}} + \right.$$
$$\left. \frac{\alpha x!}{(1+2\theta+2\lambda\theta)^{x+1}} + \frac{\alpha(1+\lambda)x!}{(1+2\theta)^{x+1}} - \frac{\alpha(2+\lambda)x!}{(1+2\theta+\lambda\theta)^{x+1}} \right)$$

$$= \frac{\theta(1+\lambda)}{\lambda^2} \left(\frac{\lambda(\alpha-1)}{(1+\theta+\lambda\theta)^{x+1}} - \frac{\lambda(\alpha-1)}{(1+\theta)^{x+1}} + \right.$$
$$\left. \frac{2\alpha}{(1+2\theta+2\lambda\theta)^{x+1}} + \frac{2\alpha(1+\lambda)}{(1+2\theta)^{x+1}} - \frac{2\alpha(2+\lambda)}{(1+2\theta+\lambda\theta)^{x+1}} \right)$$

Special Cases:

1. Equation (11) becomes the mixed Poisson Weighted Exponential Distribution (Zamani et al., 2014) when the shape parameter α=0 with the PMF given in Equation (12) as:

$$t(x) = \frac{\theta(1+\lambda)}{\lambda} \left(\frac{1}{(1+\theta)^{x+1}} - \frac{1}{(1+\theta+\lambda\theta)^{x+1}} \right) \tag{12}$$

2. When λ=1, the PMF in (11) becomes a new two-parameter mixed Poisson distribution (N2-PMPD) with PMF given in Equation (13) as:

$$t(x) = 2\theta \left(\frac{(\alpha-1)}{(1+2\theta)^{x+1}} - \frac{\alpha-1}{(1+\theta)^{x+1}} + \frac{2\alpha}{(1+4\theta)^{x+1}} \right.$$
$$\left. + \frac{4\alpha}{(1+2\theta)^{x+1}} - \frac{6\alpha}{(1+3\theta)^{x+1}} \right) \tag{13}$$

3. When λ=1 and α=0, the PMF in (11) becomes a new one-parameter mixed Poisson distribution (N1-PMPD) with PMF given in Equation (14) as:

$$t(x) = 2\theta \left(\frac{1}{(1+\theta)^{x+1}} - \frac{1}{(1+2\theta)^{x+1}} \right) \tag{14}$$

Different shapes of the PMF of the mixed Poisson QTWED are shown for different values of the parameters θ, λ, and a in Figure 3. The Figure shows that the distribution can effectively model unimodal positively skewed observations with many zeros, imitating shapes and characteristics of the QTWED. The shapes also resembles those obtained for the mixing distribution (QTWED) in Figure 1.

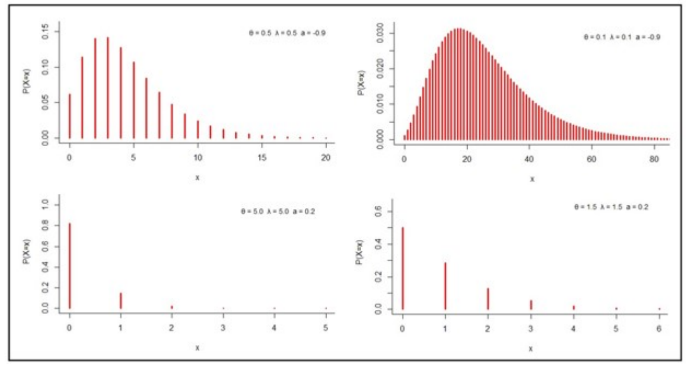

Figure 3. Shapes of the PMF of the Mixed Poisson QTWED

2.2.1 The CDF of the mixed Poisson QTWED

Proposition 4: If a discrete random variable X has the mixed Poisson QTWED, the CDF is obtained in Equation (15) as:

$$T(x) = 1 - \left(\frac{\alpha-1}{\lambda(1+\theta+\lambda\theta)^{x+1}} - \frac{(\alpha-1)(1+\lambda)}{\lambda(1+\theta)^{x+1}} + \right.$$
$$\frac{\alpha}{\lambda^2(1+2\theta+2\lambda\theta)^{x+1}} + \frac{\alpha(1+\lambda)^2}{\lambda^2(1+2\theta)^{x+1}}$$
$$\left. - \frac{2\alpha(1+\lambda)}{\lambda^2(1+2\theta+\lambda\theta)^{x+1}} \right) \tag{15}$$

Proof

$$T(x) = P(X \le x)$$
$$= 1 - P(X > x)$$
$$= 1 - \sum_{k=x+1}^{\infty} t(k)$$

$$= 1 - \sum_{k=x+1}^{\infty} \left(\frac{\theta(1+\lambda)}{\lambda^2} \left(\frac{\lambda(\alpha-1)}{(1+\theta+\lambda\theta)^{k+1}} - \frac{\lambda(\alpha-1)}{(1+\theta)^{k+1}} + \right. \right.$$
$$\left. \left. \frac{2\alpha}{(1+2\theta+2\lambda\theta)^{k+1}} + \frac{2\alpha(1+\lambda)}{(1+2\theta)^{k+1}} - \frac{2\alpha(2+\lambda)}{(1+2\theta+\lambda\theta)^{k+1}} \right) \right)$$

$$= 1 - \left(\frac{\theta(1+\lambda)}{\lambda^2} \left(\frac{\lambda(\alpha-1)}{\theta(1+\lambda)(1+\theta+\lambda\theta)^{x+1}} - \right. \right.$$
$$\frac{\lambda(\alpha-1)}{\theta(1+\theta)^{x+1}} + \frac{2\alpha}{2\theta(1+\lambda)(1+2\theta+2\lambda\theta)^{x+1}}$$
$$\left. \left. + \frac{2\alpha(1+\lambda)}{2\theta(1+2\theta)^{x+1}} - \frac{2\alpha(2+\lambda)}{\theta(2+\lambda)(1+2\theta+\lambda\theta)^{x+1}} \right) \right)$$

$$= 1 - \left(\frac{(\alpha-1)}{\lambda(1+\theta+\lambda\theta)^{x+1}} - \frac{(\alpha-1)(1+\lambda)}{\lambda(1+\theta)^{x+1}} \right.$$
$$+ \frac{\alpha}{\lambda^2(1+2\theta+2\lambda\theta)^{x+1}} + \frac{\alpha(1+\lambda)^2}{\lambda^2(1+2\theta)^{x+1}}$$
$$\left. - \frac{2\alpha(1+\lambda)}{\lambda^2(1+2\theta+\lambda\theta)^{x+1}} \right)$$

2.2.2 Reliability Measures of the mixed Poisson QTWED

The Survival function of the mixed Poisson QTWED is given in Equation (16) as:

$$S(x)=\frac{(\alpha-1)}{\lambda(1+\theta+\lambda\theta)^{x+1}}-\frac{(\alpha-1)(1+\lambda)}{\lambda(1+\theta)^{x+1}} + \frac{\alpha}{\lambda^2(1+2\theta+2\lambda\theta)^{x+1}}+\frac{\alpha(1+\lambda)^2}{\lambda^2(1+2\theta)^{x+1}} - \frac{2\alpha(1+\lambda)}{\lambda^2(1+2\theta+\lambda\theta)^{x+1}} \tag{16}$$

The hazard rate function is given in Equation (17) as:

$$h(x)=\frac{\frac{\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{(1+\theta+\lambda\theta)^{x+1}}-\frac{\lambda(\alpha-1)}{(1+\theta)^{x+1}}+\frac{2\alpha}{(1+2\theta+2\lambda\theta)^{x+1}}+\right.}{\frac{(\alpha-1)}{\lambda(1+\theta+\lambda\theta)^{x+1}}-\frac{(\alpha-1)(1+\lambda)}{\lambda(1+\theta)^{x+1}}+\frac{\alpha}{\lambda^2(1+2\theta+2\lambda\theta)^{x+1}}+}$$

$$\frac{\left.\frac{2\alpha(1+\lambda)}{(1+2\theta)^{x+1}}-\frac{2\alpha(2+\lambda)}{(1+2\theta+\lambda\theta)^{x+1}}\right)}{\frac{\alpha(1+\lambda)^2}{\lambda^2(1+2\theta)^{x+1}}-\frac{2\alpha(1+\lambda)}{\lambda^2(1+2\theta+\lambda\theta)^{x+1}}} \tag{17}$$

2.2.3 Probability Generating Function of the mixed Poisson QTWED

The Probability Generating Function (PGF) of a random variable X with the mixed Poisson QTWED is obtained in Equation (18) as:

$$P_x(s)=\int_0^\infty e^{y(s-1)}f(y)dy$$

$$=\int_0^\infty e^{y(s-1)}\frac{2\theta(1+\lambda)}{\lambda^2}\left(\lambda\frac{\alpha-1}{2}e^{-(1+\lambda)\theta y}-\frac{\lambda(\alpha-1)}{2}e^{-\theta y}+\alpha e^{-2(1+\lambda)\theta y}+\alpha(1+\lambda)e^{-2\theta y}-\alpha(2+\lambda)e^{-(2+\lambda)\theta y}\right)dy$$

$$=\frac{2\theta(1+\lambda)}{\lambda^2}\int_0^\infty\left(\lambda\frac{\alpha-1}{2}e^{-(1+\theta+\lambda\theta-s)y}-\frac{\lambda(\alpha-1)}{2}e^{-(1+\theta-s)y}+\alpha e^{-(1+2\theta+2\lambda\theta-s)y}+\alpha(1+\lambda)e^{-(1+2\theta-s)y}-\alpha(2+\lambda)e^{-(1+2\theta+\lambda\theta-s)y}\right)dy$$

$$P_x(s)=\frac{\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{(1+\theta+\lambda\theta-s)}-\frac{\lambda(\alpha-1)}{(1+\theta-s)}+\frac{2\alpha}{(1+2\theta+2\lambda\theta)}+\frac{2\alpha(1+\lambda)}{(1+2\theta-s)}-\frac{2\alpha(2+\lambda)}{(1+2\theta+\lambda\theta-s)}\right) \tag{18}$$

2.2.4 Moment Generating Function of the mixed Poisson QTWED

The MGF is obtained as by replacing s in (18) with e^t. Hence, the MGF is given in Equation (19) as:

$$M_x(t)=\frac{\theta(1+\lambda)}{\lambda^2}\left(\frac{\lambda(\alpha-1)}{(1+\theta+\lambda\theta-e^t)}-\frac{\lambda(\alpha-1)}{(1+\theta-e^t)}+\frac{2\alpha}{(1+2\theta+2\lambda\theta-e^t)}+\frac{2\alpha(1+\lambda)}{(1+2\theta-e^t)}-\frac{2\alpha(2+\lambda)}{(1+2\theta+\lambda\theta-e^t)}\right) \tag{19}$$

Using the relationship $E(X^!)=\frac{d^!MX(t)}{dt^!}|_{t=0}$ for i=1, 2, ... The first four central moments for the distribution respectively denoted by E(X), $E(X^2)$, $E(X^3)$, and $E(X^4)$ are obtained in Equations (20)–(23) as:

$$E(X)=\frac{(2-\alpha)\lambda^2+(8-3\alpha)\lambda+8-3\alpha}{2\theta(1+\lambda)(2+\lambda)} \tag{20}$$

$$E(X^2)=\frac{1}{2\theta^2(2+\lambda)^2(1+\lambda)^2}(4-3\alpha+(2-\alpha)\theta)\lambda^4+(28-20\alpha+(14-6\alpha)\theta)\lambda^3+(76-50\alpha+(36-14\alpha)\theta)\lambda+(96-60\alpha+(40-15\alpha)\theta)\lambda+(48-30\alpha+(16-6\alpha)\theta) \tag{21}$$

$$E(X^3)=\frac{\theta(1+\lambda)}{\lambda^2}\left(\frac{6(\alpha-1)}{(\lambda\theta+\theta)^4}+\frac{6\lambda(\alpha-1)}{(\lambda\theta+\theta)^3}+\frac{\lambda(\alpha-1)}{(\lambda\theta+\theta)^2}-6\lambda(\alpha-1)\left(\frac{1}{\theta^4}+\frac{1}{\theta^3}+\frac{1}{6\theta^2}\right)+\frac{12\alpha}{(2\lambda\theta+2\theta)^4}+\frac{12\alpha}{(2\lambda\theta+2\theta)^3}+\frac{2\alpha}{(2\lambda\theta+2\theta)^2}+3\alpha(1+\lambda)\left(\frac{1}{4\theta^4}+\frac{1}{2\theta^3}+\frac{1}{6\theta^2}\right)-\frac{12\alpha(2+\lambda)}{(\lambda\theta+2\theta)^4}-\frac{12\alpha(2+\lambda)}{(\lambda\theta+2\theta)^3}-\frac{12\alpha(2+\lambda)}{(\lambda\theta+2\theta)^4}\right) \tag{22}$$

$$E(X^4)=\frac{\theta(1+\lambda)}{\lambda^2}\left(\lambda(\alpha-1)\left[\frac{24}{(\lambda\theta+\theta)^5}-\frac{24}{\theta^5}+\frac{1}{(\alpha\theta+\theta)^2}-\frac{1}{\theta^2}+\frac{14}{(\alpha\theta+\theta)^3}-\frac{14}{\theta^3}-\frac{36}{\theta^4}+\frac{36}{(\lambda\theta+\theta)^4}\right]+\alpha(1+\lambda)\left[\frac{3}{2\theta^5}+\frac{1}{2\theta^2}+\frac{7}{2\theta^3}+\frac{9}{2\theta^4}\right]-\alpha(2+\lambda)\left[\frac{48}{(\lambda\theta+2\theta)^5}+\frac{72}{(\lambda\theta+2\theta)^4}+\frac{28}{(\lambda\theta+2\theta)^3}+\frac{2}{(\lambda\theta+2\theta)^2}\right]+\frac{\alpha}{(2\lambda\theta+2\theta)^2}\left[2+\frac{28}{(2\lambda\theta+2\theta)}+\frac{72}{(2\lambda\theta+2\theta)^2}+\frac{48}{(2\lambda\theta+2\theta)^3}\right]\right) \tag{23}$$

2.2.5 Measure of Skewness and Kurtosis

Given the first four moments as derived in Equation (20) – (23), expressions for both skewness and kurtosis are cumbersome but can be respectively obtained using:

$$S_k(N)=\frac{E(N^3)-3E(N^2)E(N)+2(E(N))^3}{(Var(N))^{\frac{3}{2}}}$$

$$Kurt(N)=\frac{E(N^4)-4E(N^3)E(N)+6E(N^2)(E(N))^2}{(Var(N))^2}$$

$$\frac{-3(E(N))^4}{(Var(N))^2}$$

The Index of Dispersion (DI) which assesses the degree of dispersion in observations for the distribution is defined as:

$$DI=\frac{Var(N)}{E(N)}$$

Tables 1–3 show simulated values for different parameters combinations for the Skewness, Kurtosis, and Dispersion Index of the mixed Poisson QTWED.

Remarks:

1. For fixed α and λ, both skewness and kurtosis increase while the dispersion index reduces as θ increases.
2. For fixed α and θ, skewness and kurtosis increase as λ increases. The dispersion index mostly peaked at λ=7.5.
3. For fixed λ and θ, both skewness and kurtosis increases while the dispersion index decreases as α increases.

2.2.6 Maximum Likelihood Estimation for the mixed Poisson-QTWED

Assuming x_i ,(i=1,2,...k) are random sample of size k from the mixed Poisson-QTWED with (θ,λ,α) as defined in the Equation (11), the log-likelihood function for the distribution is obtained in Equation (24) as:

$$l=\sum_{i=1}^{k}log\left(\frac{\theta(1+\alpha)}{\alpha^2}\left(\frac{\alpha(p-1)}{(1+\theta+\alpha\theta)^{ni+1}}-\frac{\alpha(p-1)}{(1+\theta)^{ni+1}}+\frac{2p}{(1+2\theta+2\theta\alpha)^{ni+1}}+\frac{2p(1+\alpha)}{(1+2\theta)^{ni+1}}-\frac{2p(2+\alpha)}{(1+2\theta+\theta\alpha)^{ni+1}}\right)\right) \tag{24}$$

The maximum likelihood estimators for (θ,λ,α) denoted with ($\bar{\theta}$, $\bar{\lambda}$, $\bar{\alpha}$) are of the non-linear equations $\frac{dl}{d\theta}$=0, $\frac{dl}{d\lambda}$=0, and $\frac{dl}{d\alpha}$=0. Due to the complex form of obtaining the necessary differential equation, we utilize the maxLik function from the R language R Core Team and Team (2021) to obtain the set of numerical solutions for the equation.

2.3 Applications

This section considers some non-life insurance datasets to assess proposed distribution along with similar count distributions. These data represent frequency of claims in Sweden, Australia, Belgium, and Turkey. The data had been used to assess new propositions by different authors (R Core Team and Team, 2021; Denuit, 1997; Meytrianti et al., 2019; Ohlsson and Johansson, 2010; Omari et al., 2018; Sarul and Sahin, 2015; Zamani et al., 2014). All the data are dispersed with positive skewness as shown in Table 4.

2.4 Competing Distributions

The performance of the new distribution is assessed on the four datasets using the following discrete distributions with PMF as presented in Table 5.

1. Poisson distribution:
2. Zero Inflated Poisson (ZIP) distribution
3. Negative Binomial (NB) distribution
4. Zero Inflated Negative Binomial (ZINB) distribution
5. Mixed Poisson Weighted Exponential Distribution (PW ED)
6. A New 2-Parameter Mixed Poisson distribution (N2-MPD) when λ=1, and
7. A New 1-Parameter Mixed Poisson distribution (N1-MPD) when λ=1 and α=0.

3. RESULTS AND DISCUSSION

Tables 6–9 show results obtained for applying the new distribution and the competing distributions to the four datasets.

Table 6 shows that the new proposition (Poisson-QTWED) provides the best fit for the first dataset with the least chi-square value (0.01) and the least -2LL (7681.24).

The mixed Poisson-QTWED also gives the best fit for the second dataset. The new 2-parameter Mixed Poisson Distribution (N2-MPD) also gives better fit to the dataset than the 2-parameter Zero Inflated Poisson (ZIP) distribution while

Table 1. Skewness for Selected Parameter for Mixed Poisson-QTWED

	λ=0.5			λ=2.5			λ=7.5		
	θ=0.1	θ=2.0	θ=10	θ=0.1	θ=2.0	θ=10	θ=0.1	θ=2.0	θ=10
α=-0.9	1.2447	1.4155	2.3405	1.4878	1.6119	2.6395	1.5761	1.7402	2.8164
α=-0.5	1.2940	1.5400	2.5127	1.5644	1.7552	2.8401	1.6793	1.9040	3.0450
α=0.0	1.4878	1.7165	2.7582	1.7907	1.9592	3.1335	1.9415	2.1440	3.3876
α=0.5	1.7572	1.8836	3.0385	2.1166	2.1586	3.4822	2.3325	2.3963	3.8115
α=0.9	1.6743	1.8932	3.2712	2.0499	2.1746	3.7891	2.3511	2.4469	4.2045

Table 2. Kurtosis for Selected Parameter for Mixed Poisson-QWTED

	λ=0.5			λ=2.5			λ=7.5		
	θ=0.1	θ=2.0	θ=10	θ=0.1	θ=2.0	θ=10	θ=0.1	θ=2.0	θ=10
α=-0.9	5.5459	5.7386	9.1477	6.6016	6.5982	10.8408	6.9290	7.1855	11.9822
α=-0.5	5.6561	6.2182	10.1424	6.8924	7.2461	12.1408	7.3562	7.9911	13.5710
α=0.0	6.4070	7.0335	11.6598	8.0150	8.3332	14.1776	8.7421	9.3815	16.1459
α=0.5	8.0091	7.9546	13.4600	10.3579	9.5862	16.7126	11.6713	11.1257	19.5399
α=0.9	8.1871	7.8407	14.7702	11.0005	9.4384	18.7126	13.1587	11.2319	22.4531

Table 3. Dispersion Index for Selected Parameter for Mixed Poisson-QWTED

	λ=0.5			λ=2.5			λ=7.5		
	θ=0.1	θ=2.0	θ=10	θ=0.1	θ=2.0	θ=10	θ=0.1	θ=2.0	θ=10
α=-0.9	8.2677	1.3634	1.0727	8.3395	1.3670	1.0734	8.9617	1.3981	1.0796
α=-0.5	9.1947	1.4097	1.0819	9.0992	1.4050	1.0810	9.7376	1.4369	1.0874
α=0.0	9.6667	1.4333	1.0867	9.4127	1.4206	1.0841	10.0712	1.4536	1.0907
α=0.5	8.8745	1.3937	1.0787	8.5034	1.3752	1.0750	9.1182	1.4059	1.0812
α=0.9	6.5946	1.2797	1.0559	6.0835	1.2542	1.0508	6.4380	1.2719	1.0544

Table 4. Data Summary

Dataset	Percentage of 0	Mean	Variance	Dispersion Index	Skewness	Kurtosis
Dataset 1	98.96	0.011	0.012	1.09	10.46	118.26
Dataset 2	93.19	0.073	0.077	1.05	4.07	18.50
Dataset 3	90.33	0.106	0.115	1.08	3.52	14.59
Dataset 5	79.01	0.265	0.334	1.26	7.71	2.56

Table 5. PMF of the Competing Distributions

Distribution	PMF
Poisson	$\frac{e^{\theta}\theta^{x}}{x!}$
ZIP	$\{\ \alpha+(1\text{-}\alpha)\mathrm{e}^{-\theta}$, x=0 $(1\text{-}\alpha)\frac{e^{-\theta}\theta^{x}}{den}$, x=1,2,3,...
NB	$\frac{(x+\lambda-1)}{x}\ \frac{\theta}{1+\theta}\ x\ \frac{1}{1+\theta}\lambda$
ZINB	$\{\ \alpha + (1\text{-}\alpha)\ \frac{1}{1+\theta}\lambda$, x= 0 $(1\text{-}\alpha)\ [\frac{x+\lambda-1}{x}\ \frac{\theta}{1+\theta}x\ \frac{1}{1+\theta}\lambda\ ]$ x=1,2,3..
PWED	$\frac{\theta(1+\lambda)}{\lambda}\ \frac{1}{(1+\theta)^{x+1}} - \frac{1}{(1+\theta+\lambda\theta)^{x+1}}$
N2-MPD	$2\theta\ \frac{(\alpha-1)}{1+2\theta} - \frac{(\alpha-1)}{1+\theta^{x+1}} + \frac{2\alpha}{4+\theta^{x+1}} + \frac{4\alpha}{2+\theta^{x+1}} - \frac{6\alpha}{3+\theta^{x+1}}$
N1-MPD	$2\theta\ \frac{1\alpha}{1+\theta^{x+1}} - \frac{1\alpha}{1+2\theta^{x+1}}$

Table 6. Assuming Different Distributions for Claims Frequency from Sweden (Dataset 1)

Obs	Freq	Poisson	ZIP	Neg. Bin	ZINB	PQTWED	PWED	N2-MPD	N1-MPD
0	63878	63854.63	63878.11	63877.73	63893.56	63880.07	63858.25	63848.59	63855.13
1	643	689.63	643.61	644.62	629.97	640.82	682.39	693.41	687.07
2	27	3.72	25.58	24.43	23.26	26.65	7.28	5.95	5.75
					$\bar{\theta}$=0.0032	$\bar{\theta}$=41.1176			
			$\bar{\theta}$=0.0795	$\bar{\theta}$=0.1536	$\bar{\alpha}$=0.9264	$\bar{\alpha}$=0.0763	$\bar{\theta}$=92.713	$\bar{\theta}$=94.7556	$\bar{\theta}$=138.5738
	Estimates	$\bar{\theta}$=0.0108	$\bar{\alpha}$=0.8642	$\bar{\lambda}$=0.9343	$\bar{\lambda}$=-40.71	$\bar{\lambda}$=-3.0010	$\bar{\lambda}$=714.615	$\bar{\alpha}$=0.7961	
	-2 LL	7744.00	7681.76	7683.00	7714.64	7681.24	7714.114	7723.736	7724.324
	Chi-Square	148.64	0.08	0.27	0.88	0.01	55.68	78.12	81.41

Table 7. Assuming Different Distributions for Claims Frequency from Australia (Dataset 2)

Obs	Freq	Poisson	ZIP	Neg. Bin	ZINB	PQTWED	PWED	N2-MPD	N1-MPD
0	63232	63091.61	63230.49	63230.60	63317.89	63232.51	63245.08	63195.49	63184.65
1	4333	4593.07	4325.83	4330.57	4252.49	4330.56	4306.47	4399.89	4418.98
2	271	167.19	286.59	276.48	261.98	274.16	284.36	246.57	239.84
3	18	4.06	12.66	17.22	21.45	17.55	18.76	13.29	11.93
4	2	0.07	0.42	1.06	1.97	1.14	1.24	0.72	0.57
					$\bar{\theta}$=0.0067	$\bar{\theta}$=14.1614			
	Estimates	$\bar{\theta}$=0.0728	$\bar{\theta}$=0.1325	$\bar{\theta}$=1.1569	$\bar{\alpha}$=0.8776	$\bar{\alpha}$=0.1836	$\bar{\theta}$=14.1595	$\bar{\theta}$=16.4989	$\bar{\theta}$=20.6170
			$\bar{\alpha}$=0.4507	$\bar{\lambda}$=0.9408	$\bar{\lambda}$=-76.06	$\bar{\lambda}$=7.1120	$\bar{\lambda}$=32.1403	$\bar{\alpha}$=0.5140	$\bar{\theta}$=20.6170
	-2 LL	36203.00	36104.40	36099.36	36211.16	36099.16	36099.82	36104.92	36108.9
	Chi-Square	177.66	9.07	0.98	2.51	0.70	1.29	7.40	12.40

Table 8. Assuming Different Distributions for Claims Frequency from Belgium (Dataset 3)

Obs	Freq	Poisson	ZIP	Neg. Bin	ZINB	PQTWED	PWED	N2-MPD	N1-MPD
0	57178	56949.76	57177.48	57188.34	57249.63	57179.28	57178.37	57138.80	57120.00
1	5167	6019.59	5584.80	5581.31	5558.90	5597.59	5601.97	5671.46	5703.79
2	446	318.14	504.87	485.28	438.37	475.70	475.68	451.34	441.60
3	50	11.21	30.43	40.47	45.91	41.96	39.44	34.52	31.32
4	8	0.30	1.38	3.30	5.40	4.00	3.25	2.66	2.14
					$\bar{\theta}$=0.0078	$\bar{\theta}$=38.1142			
	Estimates	$\bar{\theta}$=0.1057	$\bar{\theta}$=0.1808	$\bar{\theta}$=1.2791	$\bar{\alpha}$=0.8435	$\bar{\alpha}$=0.7753	$\bar{\theta}$=11.1280	$\bar{\theta}$=11.16729	$\bar{\theta}$=14.192
			$\bar{\alpha}$=0.4153	$\bar{\lambda}$=0.9237	$\bar{\lambda}$=71.13	$\bar{\lambda}$=-0.8021	$\bar{\lambda}$=4.7004	$\bar{\alpha}$=0.5483	
	-2 LL	44301.08	44150.60	44128.62	44273.14	44125.14	44127.74	44131.8	44138.26
	Chi-Square	413.84	51.55	12.33	2.44	7.46	11.64	18.25	28.64

Table 9. Assuming Different Distributions for Claims Frequency from Turkey (Dataset 4)

Obs	Freq	Poisson	ZIP	Neg. Bin	ZINB	PQTWED	PWED	N2-MPD	N1-MPD
0	8544	8292.42	8544.19	8543.47	8561.78	8537.64	8535.03	8456.65	8441.51
1	1796	2201.64	1759.23	1795.62	1807.66	1808.11	1808.98	1934.19	1955.13
2	370	292.27	430.75	375.71	331.89	369.40	373.12	350.52	349.68
3	81	25.87	70.31	78.50	81.03	77.57	76.90	60.18	57.10
4	23	1.72	8.61	16.39	22.23	16.64	15.85	10.30	8.95
					$\bar{\theta}$=0.0055	$\bar{\theta}$=3.5157			
	Estimates	$\bar{\theta}$=0.2655	$\bar{\theta}$=0.4897	$\bar{\theta}$=1.0090	$\bar{\alpha}$=0.6348	$\bar{\alpha}$=0.2636	$\bar{\theta}$=3.8520	$\bar{\theta}$=4.5848	$\bar{\theta}$=5.6519
			$\bar{\alpha}$=0.4579	$\bar{\lambda}$=0.7917	$\bar{\lambda}$=82.4300	$\bar{\lambda}$=14.2414	$\bar{\lambda}$=43.1422	$\bar{\alpha}$=0.4868	
	-2 LL	14306.32	14077.82	14059.42	14114.12	14059.92	14059.65	14082.7	14091.36
	Chi-Square	484.41	35.02	2.84	4.51	2.67	3.57	34.74	47.42

the Negative Binomial distribution provides the second best fit.

From Table 8, the Poisson-QTWED provides the least -2LL for the third dataset while its chi-square goodness of fit statistics is second to that of Zero Inflated Negative Binomial distribution. The three special form of the new proposition (PWED, N2-MPD, and N1-MPD) also give better fit to the dataset that bot Poisson and the ZIP distributions.

For the fourth dataset, Table 9 shows that the new distributions has the least chi-square goodness of fit statistics (2.67) and its -2LL is not distinguishable from the best two (obtained for Negative Binomial and PWED distributions).

4. CONCLUSION

Using the quadratic transmutation map on the weighted exponential distribution to obtain a new mixing distribution, this paper introduce a new three-parameter mixed Poisson distribution named the Poisson Quadratic Transmuted Weighted Exponential Distribution (Poisson-QTWED). Basic statistical properties of both mixing distribution and the mixed Poisson distribution are obtained. Shapes of the mixing distribution is observed to be similar to the shapes of the mixed Poisson distribution. For various parameter combinations, skewness, kurtosis, and the dispersion index of the new distribution are assessed in a simulation study. Using the maximum likelihood estimation, the distribution is assessed on four dispersed claim frequency with high percentage of zero counts from different countries. Comparisons are made with the Poisson and negative binomial distributions (along with their respective zero-inflated forms). Results show that the new distribution gives satisfactory fit to the datasets and hence can serve as a better alternative to model dispersed observations with many zero.

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