Science and Technology Indonesia
e-ISSN:2580-4391 *p*-ISSN:2580-4405
Vol. 10, No. 1, January 2025

Research Paper

Relation Between the First Zagreb and Greatest Common Divisor Degree Energies of Commuting Graph for Dihedral Groups

Mamika Ujianita Romdhini[1,2], Athirah Nawawi[1*]

[1]*Department of Mathematics and Statistics, Faculty of Science, Universiti Putra Malaysia, Serdang, Selangor, 43400, Malaysia*
[2]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, University of Mataram, Mataram, 83125, Indonesia*
***Corresponding author**: athirah@upm.edu.my

Abstract

The commuting graph for a finite group G, Γ_G, has a set of vertices $G \setminus Z(G)$, where $Z(G)$ is the center of G, and v_p,$v_q \in G \setminus Z(G)$ in which $v_p \neq v_q$, are adjacent whenever $v_p v_q = v_q v_p$. The entries of the first Zagreb matrix (Z_1) of Γ_G are either the summation of the degrees of two adjacent vertices, or zero for non-adjacent vertices and also for the diagonal entries. Meanwhile, the entries of the greatest common divisor degree matrix ($GCDD$) of Γ_G are the greatest common divisor of the degrees of two adjacent vertices and zero otherwise. The Z_1-energy is determined by the sum of absolute eigenvalues of the corresponding Z_1-matrix, whereas $GCDD$-energy is the sum of absolute eigenvalues of the $GCDD$-matrix. In this study, we find the spectral radius and the energies of Γ_G for dihedral groups of order $2n$, D_{2n}, associated with Z_1- and $GCDD$-matrices. It is found that Z_1-energy is equal to twice $GCDD$-energy, whereas $GCDD$-energy is similar to maximum and minimum degree energies that were reported earlier in previous literature.

Keywords

First Zagreb Matrix, Greatest Common Divisor Degree Matrix, Energy of a Graph, Commuting Graph, Dihedral Group

Received: 5 December 2023, Accepted: 25 August 2024
`https://doi.org/10.26554/sti.2025.10.1.1-8`

1. INTRODUCTION

A central idea in spectral graph theory is to associate matrices with graphs, notably, the adjacency matrix by knowing the corresponding eigenvalues. Converting a chemical molecule into a graph and measuring π-electrons energy in the molecule, Gutman (1978) discovered the energy of a finite graph as the summation of absolute eigenvalues. The graph energy bounds can be seen in Filipovski and Jajcay (2021). For simple graphs, Das et al. (2018) supplied the bounds of energy corresponding to the degree-based matrices. Moreover, the fact that the energy is never equal to an odd integer (Bapat and Pati, 2004) or its square root (Pirzada and Gutman, 2008) is very important in this research topic.

Furthermore, the graph's spectra also emphasize the discussion of spectral radius. As defined by (Horn and Johnson, 1985), a spectral radius for a graph Γ_G is the greatest of the absolute eigenvalues associated with the graph matrices. Several graphs have been studied for the spectral radius problem, for instance in Ganie and Shang (2022). According to Chattopadhyay et al. (2018), the spectral radius can be related to power graphs for dihedral groups. In addition, the graphs on n vertices with an energy more than the energy of the complete graph can be categorized as hyperenergetic, or in other words when it is more than $2(n-1)$ (Li et al., 2012).

This paper discusses the simple graph whose vertices are group elements and whose edges are defined by the commutativity between elements, which is called the commuting graph and is denoted by Γ_G. The commuting graph for a group G has the vertex set $G \setminus Z(G)$ and $v_r, v_s \in G \setminus Z(G)$ are adjacent whenever $v_r v_s = v_s v_r$ (Brauer and Fowler, 1955).

Significant progress in recent years has been in the area of algebraic graph theory specifically on commuting graphs. For example, discussion on the spectral and energy of Γ_G for the dihedral groups based on degree exponent sum (Romdhini et al., 2022), maximum and minimum degree (Romdhini and Nawawi, 2022) matrices. Meanwhile, the greatest common divisor degree energy of some simple graphs was investigated by Ramkumar and Nagarajan (2017) and its bounds (Ramkumar and Nagarajan, 2018). Furthermore, Rad et al. (2018) explored the bounds of Zagreb energy for simple graphs and gave the definition of the first Zagreb and second Zagreb matrices of order $n \times n$ associated with graphs. Later, Bhat and Shetty (2024) showed the first Zagreb energy for k-half graph and discussed its applications in their subsequent work (Shetty and Bhat, 2024). Meanwhile, the discussion on hypergraph can be seen in Sharmila et al. (2023) and the connection between Laplacian energy and the first Zagreb index has been discussed in Hameed et al. (2022). Shao et al. (2021) presented four types of degree-based energies: Zagreb, harmonic, geometric-arithmetic, and sum-connectivity energies for trees. In addition, Jahanbani et al. (2022) presented the new bounds of Zagreb energy of a graph.

Throughout this paper, we focus on the dihedral group of order $2n$, $D_{2n} = \langle a, b : a^n = b^2 = e, bab = a^{-1}\rangle$. The centre of D_{2n} is either $Z(D_{2n}) = \{e\}$ for odd n, or $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$ if n is even (Aschbacher, 2000). The centralizer of a^i in D_{2n} is $C_{D_{2n}}(a^i) = \{a^j : 1 \leq j \leq n\}$ whereas for a^ib is either $C_{D_{2n}}(a^ib) = \{e, a^ib\}$ for odd n, or $C_{D_{2}n}(a^ib) = \{e, a^{\frac{n}{2}}, a^ib, a^{\frac{n}{2+i}}b\}$ for even n. The discussion of this paper is devoted to the commuting graph for D_{2n}, where $n \geq 3$.

This paper first discusses several existing results that aid the computation of our findings, then goes on to Section 3 which provides the formulas of energy of Γ_G for dihedral groups D_{2n}, where $n \geq 3$, associated with the first Zagreb and greatest common divisor degree matrices. Section 4 further discusses the relationship between Z_1- and $GCDD-$ energies of Γ_G for D_{2n}, $n \geq 3$, and also supplies some properties.

2. PRELIMINARIES

Let d_{v_p} be the degree of vertex v_p.

Definition 2.1. (Rad et al., 2018) The $n \times n$ first Zagreb matrix of Γ_G is given by $Z_1(\Gamma_G) = (Z_{1_{pq}})$ in which (p, q)-th entry is

$$Z_{1_{pq}} = \begin{cases} d_{v_p} + d_{v_p}, & \text{if } v_p \neq v_q \text{ and they are adjacency} \\ 0, & \text{otherwise.} \end{cases}$$

Definition 2.2. (Ramkumar and Nagarajan, 2017) The $n \times n$ greatest common divisor degree matrix of Γ_G is given by $GCDD(\Gamma_G) = (gcdd_{pq})$ in which (p, q)-th entry is

$$gcdd_{pq} = \begin{cases} gcd(d_{v_p}, d_{v_p}), & \text{if } v_p \neq v_q \text{ and they are adjac} \\ & \text{-ency} \\ 0, & \text{otherwise.} \end{cases}$$

The following are the essential results to formulate the characteristic polynomials of Γ_G.

Lemma 2.3 (Ramane and Shinde, 2017) If r, s, t, and u are real numbers, and I_n is an $n \times n$ identity matrix and $n \times n$ matrix J_n in which all entries are 1, then

$$\begin{vmatrix} (\lambda + r)I_{n_1} - rJ_{n_1} & -tJ_{n_1 \times n_2} \\ -uJ_{n_2 \times n_1} & (\lambda + s)I_{n_2} - sJ_{n_2} \end{vmatrix}$$

can be formulated simply as

$$(\lambda + r)^{n_1 - 1}(\lambda + s)^{n_2 - 1}((\lambda - (n_1 - 1)r)(\lambda - (n_2 - 1)s) - n_1 n_2 tu),$$

with $1 \leq n_1, n_2 \leq n$ and $n_1 + n_2 = n$.

Theorem 2.4. (Gantmacher, 1959) If a square matrix $M = \begin{bmatrix} M_1 & M_2 \\ M_3 & M_4 \end{bmatrix}$ can be partitioned into submatrices M_1, M_2, M_3, M_4, where $|M_1| \neq 0$, then

$$\begin{aligned} |M| &= \begin{vmatrix} M_1 & M_2 \\ 0 & M_4 - M_3M_1^{-1}M_2 \end{vmatrix} \\ &= |M_1| \left|M_4 - M_3M_1^{-1}M_2\right|. \end{aligned}$$

Lemma 2.5. (Brauer and Fowler, 1955) Let K_n represent the complete graph with n vertices. $(J - I)_n$ is the adjacency matrix of K_n with its spectrum $\{n - 1)^1, (-1)^{n-1}\}$.

The following results focus on the vertex degree of Γ_G for $G = D_{2n} \setminus Z(D_{2n}) = G_1 \cup G_2$, where $G_1 = \{a^i : 1 \leq i \leq n\} \setminus Z(D_{2n})$ and $G_2 = \{a^ib : 1 \leq i \leq n\}$.

Theorem 2.6. (Romdhini et al., 2022) In Γ_G for $G = G_1 \cup G_2$, the degree of vertex

1. a^i in Γ_G is $d_{a_i} = \begin{cases} n - 3 & \text{for odd } n \text{ and} \\ n - 3 & \text{for even } n \end{cases}$
2. a^ib in Γ_G is $d_{a_ib} = \begin{cases} 0 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

Theorem 2.7. (Romdhini et al., 2022) In Γ_G for D_{2n},

1. if $G = G_1$ and $m = |G_1|$, then $\Gamma_G \cong K_m$,
2. if $G = G_1$ then $\Gamma_G \cong \begin{cases} \bar{K}_n & \text{if } n \text{ is odd} \\ 1 - \text{regular graph} & \text{if } n \text{ is even} \end{cases}$

We denote the energy of Γ_G as $E(\Gamma_G)$. Graphs with n vertices having energy greater than $E(K_n)$ (or $> 2(n - 1)$) can be classified as hyperenergetic (Li et al., 2012). Since Γ_G contains $2n - 1$ and $2n - 2$ vertices for odd and even n, respectively, hence Γ_G associated with the $GCDD$ and Z_1-matrices satisfy the following adequate conditions to be categorized as hyperenergetic graphs:

$$E(\Gamma_G) > \begin{cases} 4(n - 1), & \text{for odd } n \\ 4(n - 1) - 2 & \text{for even } n \end{cases}$$

3. RESULTS AND DISCUSSION

The following theorem must be proved in order to formulate a matrix's characteristic polynomial. Additionally, we simplify the determinants of the characteristic formula of Γ_G by using row and column operations. As a result, we define the subsequent notations: (i) the i-th row is called R_i; (ii) the updated i-th row is represented by R_i'; (iii) the i-th column is called C_i; and (iv) the updated i-th column is represented by C_i'.

Theorem 3.1. If r is a real number, n is even, and $I_{\frac{n}{2}}$ is an $\frac{n}{2} \times \frac{n}{2}$ identity matrix, then the characteristic polynomial of an $n \times n$ matrix

$$M = \begin{bmatrix} 0_{\frac{n}{2}} & rI_{\frac{n}{2}} \\ rI_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix},$$

can be expressed as follows

$$P_M(\lambda) = (\lambda + r)^{\frac{n}{2}}(\lambda - r)^{\frac{n}{2}}.$$

Proof. Let r be a real number. The determinant $|\lambda I_n - M|$ is the characteristic formula of M as given below

$$P_M(\lambda) = \begin{vmatrix} \lambda I_{\frac{n}{2}} & -rI_{\frac{n}{2}} \\ -rI_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix},$$

In order to find the determinant, we have to simplify $P_M(\lambda)$ as an upper triangular matrix with row and column operations as follows:
Step 1: Replace $R_{\frac{n}{2}+i}$by $R'_{\frac{n}{2}+i} = R_{\frac{n}{2}+i} - R_i$, for all $1 \le i \le \frac{n}{2}$. Hence, we derive that $P_M(\lambda)$ becomes:

$$P_M(\lambda) = \begin{vmatrix} \lambda I_{\frac{n}{2}} & -rI_{\frac{n}{2}} \\ -(\lambda - r)I_{\frac{n}{2}} & (\lambda + r)I_{\frac{n}{2}} \end{vmatrix},$$

Step 2: Substitute C_i by $C'_i = C_i + C_{\frac{n}{2}+1}$, where $1 \le i \le \frac{n}{2}$. Consequently, we obtain $P_M(\lambda)$ as an upper triangular, so that

$$P_M(\lambda) = \begin{vmatrix} (\lambda - r)I_{\frac{n}{2}} & -rI_{\frac{n}{2}} \\ 0_{\frac{n}{2}} & (\lambda + r)I_{\frac{n}{2}} \end{vmatrix},$$

Knowing that the determinant is the product of the entries of the main diagonal, it can be immediately concluded that

$$P_M(\lambda) = |(\lambda - r)I_{\frac{n}{2}}||(\lambda + r)I_{\frac{n}{2}}| = (\lambda + r)^{\frac{n}{2}}(\lambda - r)^{\frac{n}{2}}.$$

3.1 Zagreb Energy

Theorem 3.2. Let $E_{Z_1}(\Gamma_G)$ be the first Zagreb energy of Γ_G.

1. If $G = G_1$, then $E_{Z_1}(\Gamma_G) \begin{cases} 4(n-2)^2 & \text{for odd } n \\ 4(n-2)^2 & \text{for even } n \end{cases}$
2. If $G = G_2$, then $E_{Z_1}(\Gamma_G) \begin{cases} 0 & \text{for odd } n \\ 2n & \text{for even } n \end{cases}$

Proof.

1. When n is odd and $G = G_1$, by Theorem 2.7 (1), one can see that $\Gamma_G \cong K_m$, where $m = |G_1|$, excluding e in $Z(D_{2n})$. In addition, by Theorem 2.6 (1), every vertex of Γ_G has degree $n-2$. By Definition 2.1, the first Zagreb matrix of order $(n-1)\times(n-1)$ of Γ_G in which its entries are either (p,q)-th entry is $z_{1_{pq}} = (n-2)+(n-2) = 2(n-2)$ for distinct adjacent v_p and v_q, or, zero otherwise, is given below

$$Z_1(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{array} \begin{bmatrix} a & a^2 & \cdots & a^{n-1} \\ 0 & 2(n-2) & \cdots & 2(n-2) \\ 2(n-2) & 0 & \cdots & 2(n-2) \\ \vdots & \vdots & \ddots & \vdots \\ 2(n-2) & 2(n-2) & \cdots & 0 \end{bmatrix}$$

$$= 2(n-2)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix} = 2(n-2)\cdot A(K_{n-1})$$

where $A(K_{n-1})$ is the adjacency matrix of $K_(n-1)$. By Lemma 2.5, $E_A(K_{n-1})$ is $|n-2|+(n-2)|-1| = 2(n-2)$, then the first Zagreb energy of Γ_G will be $2(n-2)\cdot 2(n-2) = 4(n-2)^2$.

Considering even n and $G = G_1$, by Theorem 2.7 (1) we have $\Gamma_G \cong K_m$ for $m = |G_1| = n-2$, excluding e and $a^{\frac{n}{2}}$ in $Z(D_{2n})$. Consequently, each vertex has a degree $n-3$. Now using Definition 2.1, we then obtain an $(n-2)\times(n-2)$ matrix $Z_1(\Gamma_G) = (Z_{1_{pq}})$ in which (p,q)-th entry is $Z_{1_{pq}} = (n-3)+(n-3) = 2(n-3)$, for adjacent v_p and v_q and zero otherwise, and it is written as follows:

$$Z_1(\Gamma_G) = \begin{array}{c|cccccc} & a & \cdots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}+1} & \cdots & a^{n-1} \\ \hline a & 0 & \cdots & 2(n-3) & 2(n-3) & \cdots & 2(n-3) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a^{\frac{n}{2}-1} & 2(n-3) & \cdots & 0 & 2(n-3) & \cdots & 2(n-3) \\ a^{\frac{n}{2}+1} & 2(n-3) & \cdots & 2(n-3) & 0 & \cdots & 2(n-3) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & 2(n-3) & \cdots & 2(n-3) & 2(n-3) & \cdots & 0 \end{array}$$

$$= 2(n-3)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \ddots & \vdots & \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

Based on Lemma 2.5, $E_A(K_(n-2)) = 2(n-3)$, then $E_{Z_1}(\Gamma_G)$ will be $2(n-3)\cdot 2(n-3) = 4(n-3)^2$.

2. When n is odd and $G = G_2$, by Theorem 2.7, $\Gamma_G \cong \bar{K}_n$, in which every vertex has a degree zero. By Definition 2.1, it is clear that $n\times n$ matrix $Z_1(\Gamma_G) = [0]$. The characteristic polynomial of Γ_G is $det(\lambda I_n - [0])$ and the roots are zero with multiplicity n. It implies $E_{Z_1}(\Gamma_G) = n\cdot 0 = 0$.
Considering n is even and $G = G_2$, from Theorem 2.7, Γ_G is a 1-regular graph. In other words, there is only one edge between a^ib and $a^{\frac{n}{d2+1}}b$. This implies that an $n\times n$ matrix $Z_1(\Gamma_G)$ is as follows:

$$Z_1(\Gamma_G) = \begin{array}{c|cccccccc} & b & ab & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \cdots & a^{n-1}b \\ \hline b & 0 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 \\ ab & 0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{\frac{n}{2}-1}b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 2 \\ a^{\frac{n}{2}}b & 2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a^{\frac{n}{2}+1}b & 0 & 2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & 0 & 0 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{array}$$

$$= \begin{bmatrix} 0_{\frac{n}{2}} & 2I_{\frac{n}{2}} \\ 2I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}$$

Following Theorem 3.1 with $r = 2$, hence the characteristic formula of $Z_1(\Gamma_G)$ is

$$P_{Z_1}(\Gamma_G)(\lambda) = (\lambda + 2)^{\frac{n}{2}}(\lambda - 2)^{\frac{n}{2}}.$$

Therefore, we can show the first Zagreb energy of Γ_G as follows:

$$E_{Z_1}(\Gamma_G) = (\frac{n}{2})|2| + (\frac{n}{2})|-2| = 2n.$$

Theorem 3.3. In Γ_G where $G = G_1 \cup G_2 \subset D_{2n}$,

1. for odd n, $P_{Z_1}(\Gamma_G)(\lambda) = (\lambda + 2(n-2))^{n-2}\lambda^n(\lambda - 2(n-2)^2)$, and
2. for even n, $P_{Z_1}(\Gamma_G)(\lambda) = (\lambda + 2(n-3))^{n-3}(\lambda - 2(n-3)^2)(\lambda+2)^{\frac{n}{2}}(\lambda-2)^{\frac{n}{2}}$.

Proof.

1. According to Theorem 2.6, when n is odd, $d_{a^i} = n-2$ and $d_{a^ib} = 0$, for every $1 \leq i \leq n$. Now since $Z(D_{2n}) = \{e\}$ and $G = G_1 \cup G_2$, Γ_G has $2n-1$ vertices that consist of $n-1$ vertices of a^i, for every $1 \leq i \leq n-1$, and n vertices from a^ib, for every $1 \leq i \leq n$. Hence by using Definition 2.1, we derive matrices of size $(2n-1)\times(2n-1)$:

$$Z_1(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccc} a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \end{array} \\ \begin{bmatrix} 0 & 2(n-2) & \cdots & 2(n-2) & 0 & 0 & \cdots & 0 \\ 2(n-2) & 0 & \cdots & 2(n-2) & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-2) & 2(n-2) & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}.$$

   $Z_1(\Gamma_G)$ can be partitioned into four blocks as follows:

$$Z_1(\Gamma_G) = \begin{bmatrix} 2(n-2)(J-I)_{n-1} & 0_{(n-1)} \times n \\ 0_n \times (n-1) & 0_n \end{bmatrix}.$$

   The characteristic formula of $Z_1(\Gamma_G)$ is

$$P_{Z_1(\Gamma_G)}(\lambda) = |\lambda I_{2n-1} - Z_1(\Gamma_G)| = \begin{vmatrix} (\lambda + 2(n-2))I_{n-1} - 2(n-2)J_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

   Using Lemma 2.3, with $r = 2(n-2)$, $s = t = u = 0$, $n_1 = n-1$ and $n_2 = n$, we get
2. Considering even n, consequently, by Theorem 2.6, we provide $d_{a^i} = n-3$ and $d_{a^ib} = 1$, for every $1 \leq i \leq n$. Now since $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, then Γ_G contains $2n-2$ vertices, where $G = G_1 \cup G_2$. It contains $n-2$ vertices of a^i, with $\leq i < \frac{n}{2}$, $\frac{n}{2} < i \leq n$, and n vertices of a^ib, where $1 \leq i \leq n$. Again, by Definition 2.1, matrix $Z_1(\Gamma_G)$ of size $(2n-2)\times(2n-2)$ is
   In other words,

$$Z_1(\Gamma_G) = \begin{bmatrix} 2(n-3)(J-I)_{n-2} & 0_{n-2)\times\frac{n}{2}} & 0_{n-2)\times\frac{n}{2}} \\ 0_{n-2)\times\frac{n}{2}} & 0_{\frac{n}{2}} & 2I_{\frac{n}{2}} \\ 0_{n-2)\times\frac{n}{2}} & 2I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}$$

   Then $P_{Z_1(\Gamma_G)}(\lambda)$ is
   $P_{Z_1(\Gamma_G)}(\lambda) = |\lambda I_{2n-2} - Z_1(\Gamma_G)|$

$$\begin{vmatrix} (\lambda + 2(n-3))I_{n-2} - 2(n-3)J_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{(n-2)\times\frac{n}{2}} & \lambda I_{\frac{n}{2}} & -2I_{\frac{n}{2}} \\ 0_{(n-2)\times\frac{n}{2}} & -2I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{vmatrix}$$

   According to Theorem 2.4 with $M_1 = (\lambda + 2(n-3))I_{n-2} - 2(n-3)J_{n-2}$, $M_2 = 0_{(n-2)\times n}$, $M_3 = 0_{n\times(n-2)}$, and $M_4 = \begin{bmatrix} \lambda I_{n/2} & -2I_{n/2} \\ -2I_{n/2} & \lambda I_{\frac{n}{2}} \end{bmatrix}$ then we get the form $P_(Z_1(\Gamma_G))(\lambda) = \begin{vmatrix} M_1 & M_2 \\ 0 & M_4 - M_3M_1^{-1}M_2 \end{vmatrix} = |M_1|\left|M_4 - M_3M_1^{-1}M_2\right|$ (since $M_3 = 0$). Now we need to obtain $|M_1|$. By Lemma 2.3 with $r = s = t = u = 2(n-3)$, and $n_1 = n_2 = \frac{(n-2)}{2}$, we obtain

$$|M_1| = (\lambda + 2(n-3))^{n-3}(\lambda - 2(n-3)^2).$$

   Meanwhile for $|M_4|$, by the same argument of the proof of Theorem 3.2 (2) for even n, then

$$|M_4| = (\lambda+2)^{\frac{n}{2}}(\lambda-2)^{\frac{n}{2}}.$$

   Therefore,

$$P_(Z_1(\Gamma_G))(\lambda) = (\lambda + 2(n-3))^{n-3}(\lambda - 2(n-3)^2)$$
$$(\lambda+2)^{\frac{n}{2}}(\lambda-2)^{\frac{n}{2}}.$$

Theorem 3.4. In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, the Z_1-spectral radius of Γ_G is

$$\rho_{Z_1(\Gamma_G)} = \begin{cases} (2(n-2)^2, & \text{for odd } n \\ 2(n-3)^2, & \text{for even } n. \end{cases}$$

Proof.

1. It is clear from Theorem 3.3 (1) that three eigenvalues can be obtained from $P_{Z_1(\Gamma_G)}(\lambda)$, for the odd $n.\lambda_1 = 0$ of multiplicity n, $\lambda_2 = -2(n-2)$ of multiplicity $n-2$ and a single $\lambda_3 = 2(n-2)^2$ are its eigenvalues. Hence,

$$Spec(\Gamma_G) = \{(2(n-2)^2)^1, (0)^n, (-2(n-2))^{n-2}\}. \tag{1}$$

   The largest modulus of λ_i for $i = 1, 2, 3$ is Z_1-spectral radius of Γ_G as follows

$$\rho_{Z_1(\Gamma_G)} = 2(n-2)^2.$$

2. By Theorem 3.3 (2), it is apparent that there are four eigenvalues that can be obtained for the even n case. The first is $\lambda_1 = -2(n-3)$ with multiplicity n-3, the other two eigenvalues are $\lambda_2, 3 = \pm 2$ of multiplicity $\frac{n}{2}$, respectively. The last eigenvalue is a single $\lambda_4 = 2(n-3)^2$. The spectrum of Γ_G becomes

$$Spec(\Gamma_G) = \{(2(n-3)^2)^1, (2)^{\frac{n}{2}}, (-2)^{\frac{n}{2}},$$
$$(-2(n-3))^{n-3}\}.$$

   By choosing the largest modulus eigenvalue we can obtain the Z_1-spectral radius of Γ_G as follows

$$\rho_{Z_1(\Gamma_G)} = 2(n-3)^2.$$

$$Z_1(\Gamma_G) = \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \\ b \\ ab \\ \vdots \\ a^{\frac{n}{2}-1}b \\ a^{\frac{n}{2}}b \\ a^{\frac{n}{2}+1}b \\ \vdots \\ a^{n-1}b \end{array} \begin{array}{ccccccccccccc} a & a^2 & \cdots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}} & a^{\frac{n}{2}+1} & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b & \\ \end{array}$$

$$\begin{bmatrix} 0 & 2(n-3) & \cdots & 2(n-3) & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 2(n-3) & 0 & \cdots & 2(n-3) & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 2(n-3) & 2(n-3) & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 2 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 2 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 2 & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} 0_{\frac{n}{2}} & 2I_{\frac{n}{2}} \\ 2I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}$$

Theorem 3.5. In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, the first Zagreb energy of Γ_G is

$$E_{Z_1}(\Gamma_G) = \begin{cases} (4(n-2)^2, & \text{for odd } n \\ 4(n-3)^2 + 2n, & \text{for even } n. \end{cases}$$

Proof.

1. The first Zagreb energy for Γ_G can be found by calculating the sum of the modulus of eigenvalues from $Spec(\Gamma_G)$

$$E_{Z_1}(\Gamma_G) = (n-2)|-2(n-2)| + (n)|0| + |2(n-2)^2| = 4(n-2)^2.$$

2. Similarly, for even n, we can obtain $E_{Z_1}(\Gamma_G)$ using $Spec$ (Γ_G),

$$E_{Z_1}(\Gamma_G) = (n-3)|-2(n-3)| + |2(n-3)^2| + (\frac{n}{2})|-2| + (\frac{n}{2})|2| = 4(n-3)^2 + 2n.$$

3.2 Greatest Common Divisor Degree Energy

Theorem 3.6. Let $GCDD(\Gamma_G)$ be the greatest common divisor degree matrix of Γ_G.

1. If $G = G_1$, then $E_{GCDD}(\Gamma_G) = \begin{cases} (2(n-2)^2, & \text{for odd } n \\ 2(n-3)^2, & \text{for even } n. \end{cases}$
2. If $G = G_2$, then $E_{GCDD}(\Gamma_G) = \begin{cases} 0, & \text{for odd } n \\ n, & \text{for even } n. \end{cases}$

Proof.

1. When n is odd and $G = G_1$. It follows by Theorem 2.7 (1), $\Gamma_G \cong K_m$, where $m = |G_1|$. Consequently, every vertex of Γ_G will have a degree of $n-2$. The next step is to construct an $(n-1) \times (n-1)$ greatest common divisor degree matrix of Γ_G using Definition 2.2, $GCDD(\Gamma_G) = Z_{1_{pq}}$ in which (p, q)-th entry is $gcdd_{pq} = gcd(n-2, n-2) = n-2$ for distinct adjacent v_p and v_q, or, zero otherwise:

$$GCDD(\Gamma_G)$$

$$= \begin{array}{c} \\ a \\ a^2 \\ \vdots \\ a^{n-1} \end{array} \begin{array}{c} \begin{array}{cccc} a & a^2 & \cdots & a^{n-1} \end{array} \\ \begin{bmatrix} 0 & n-2 & \cdots & n-2 \\ n-2 & 0 & \cdots & n-2 \\ \vdots & \vdots & \ddots & \vdots \\ n-2 & n-2 & \cdots & 0 \end{bmatrix} \end{array}$$

$$= (n-2) \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

$$= (n-2) \cdot A(K_{n-1}),$$

where $A(K_(n-1))$ is the adjacency matrix of $K_(n-1)$. In other words, $GCDD(\Gamma_G)$ is the product of $n-2$ and $A(K_{n-1})$. Again, by Lemma 2.5, $E_A(K_{n-1})$ is $|n-2| + (n-2)|-1| = 2(n-2)$, consequently the $GCDD$-energy of Γ_G is $(n-2) \cdot 2(n-2) = 2(n-2)^2$.

Considering even n and $G = G_1$, as shown in Theorem 2.7 (1), Γ_G is isomorphism to K_m with $m = |G_1| = n-2$. This implies that each vertex has a degree $n-3$. Now from Definition 2.1, $GCDD(\Gamma_G)$ can then be constructed as an $(n-2) \times (n-2)$ matrix, which entries are either $gcdd_{pq} = gcd(n-3, n-3) = n-3$, for adjacent v_p and v_q, or zero otherwise:

$Z_1(\Gamma_G) =$

$$\begin{array}{c} \\ a \\ \vdots \\ a^{\frac{n}{2}-1} \\ a^{\frac{n}{2}+1} \\ \vdots \\ a^{n-1} \end{array} \begin{array}{c} \begin{array}{cccccc} a & \cdots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}+1} & \cdots & a^{n-1} \end{array} \\ \begin{bmatrix} 0 & \cdots & n-3 & n-3 & \cdots & n-3 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ n-3 & \cdots & 0 & n-3 & \cdots & n-3 \\ n-3 & \cdots & n-3 & 0 & \cdots & n-3 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ n-3 & \cdots & n-3 & n-3 & \cdots & 0 \end{bmatrix} \end{array}$$

$$= (n-3)\begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 0 \end{bmatrix}$$

$= (n-3) \cdot A$,

According to Lemma 2.5, $E_A(2) = 2(n-3)$, then the greatest common divisor degree energy of Γ_G, $E_{GCDD}(\Gamma_G)$ is $(n-3) \cdot 2(n-3) = 2(n-3)^2$.

2. Considering odd n and $G = G_2$. Theorem 2.7 (2) confirms that $\Gamma_G \cong \bar{K}_n$ which then implies the degree of all vertices is zero. Thus, we have an $n \times n$ matrix $GCDD(\Gamma_G) = [0]$

   $E_{GCDD}(\Gamma_G) = 0$.

   When n is even and $G = G_2$. Based on Theorem 2.7 (2), Γ_G is isomorphic to a regular graph of degree one, which has only an edge between $a^i b$ and $a^{\frac{n}{2}+1} b$. Then following Definition 2.2, we have an $n \times n$ matrix $GCDD(\Gamma_G) = [gcdd_{pq}]$ in which (p, q)-th entry is $gcdd_{pq} = gcd(1, 1) = 1$, for adjacent v_p and v_q, and zero otherwise. Thus, we have

$$GCDD(\Gamma_G) = \begin{array}{c|cccccccc} & b & ab & \cdots & a^{\frac{n}{2}-1}b & a^{\frac{n}{2}}b & a^{\frac{n}{2}+1}b & \cdots & a^{n-1}b \\ \hline b & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ ab & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{\frac{n}{2}-1}b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ a^{\frac{n}{2}}b & & & & & & & & \\ a^{\frac{n}{2}+1}b & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array}.$$

$$= \begin{bmatrix} 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}$$

Following Theorem 3.1 with $r = 1$, $P_{GCDD(\Gamma_G)}(\lambda)$ is

$$P_{GCDD(\Gamma_G)}(\lambda) = (\lambda+1)^{\frac{n}{2}}(\lambda-1)^{\frac{n}{2}}$$

Therefore, we can show the $GCDD$-energy of Γ_G as the following:

$$E_{Z_1}(\Gamma_G) = \frac{n}{2}|1| + \frac{n}{2}|-1| = n$$

Theorem 3.7. In Γ_G where $G_1 \cup G_2 \subset D_{2_n}$,

1. for odd $nP_{GCDD(\Gamma_G)}(\lambda) = (\lambda+n-2)^{n-2}\lambda^n(\lambda-(n-2)^2)$, and
2. for even $nP_{GCDD(\Gamma_G)}(\lambda) = (\lambda+n-3)^{n-3}(\lambda-(n-3)^2)(\lambda+1)^{\frac{n}{2}}(\lambda-1)^{\frac{n}{2}}$.

Proof.

1. For the odd case, we know from Theorem 2.6 that and , where . Since , then we have vertices for , where . Then by using Definition 2.2, for two distinct adjacent vertices, the entries of are , and zero otherwise, which implies

$GCDD(\Gamma_G) =$

$$\begin{array}{c|cccccccc} & a & a^2 & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \\ \hline a & 0 & n-2 & \cdots & n-2 & 0 & 0 & \cdots & 0 \\ a^2 & n-2 & 0 & \cdots & n-2 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & n-2 & n-2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ ab & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}.$$

$GCDD(\Gamma_G)$ can be partitioned into four blocks as follows:

$$GCDD(\Gamma_G) = \begin{bmatrix} (n-2)(J-I)_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & 0_n \end{bmatrix}.$$

Here the characteristic formula of $GCDD(\Gamma_G)$ is

$$P_{GCDD}(\Gamma_G)(\lambda) = |\lambda I_{2n-1} - GCDD(\Gamma_G)|$$

$$= \begin{vmatrix} (\lambda+n-2)I_{n-1} - (n-2)J_{n-1} & 0_{(n-1)\times n} \\ 0_{n\times(n-1)} & \lambda I_n \end{vmatrix}.$$

Using Lemma 2.3, with $r = n-2$, s= $t = u = 0$, $n_1 = n-1$ and $n_2 = n$, we get

$$P_{GCDD}(\Gamma_G)(\lambda) = (\lambda+n-2)^{n-2}\lambda^n(\lambda-(n-2)^2).$$

2. The combination of Theorem 2.6 and $GCDD$-matrix definition, we obtain that $d_{a^i} = n-3$ and $d_{a^i b} = 1$, for every $1 \le i \le n$, where n is even. For $G = G_1 \cup G_2$, there are $2n-2$ vertices for Γ_G, due to $Z(D_{2n}) = \{e, a^{\frac{n}{2}}\}$, then there are $2n-2$ vertices for Γ_G. It contains $n-2$ vertices of a^i, with $1 \le i < \frac{n}{2}$, $\frac{n}{2} < i \le n$, and n vertices of $a^i b$, with $1 \le i \le n$. By Definition 2.1, $GCDD(\Gamma_G)$ can be constructed as the following

   In other words, it can be written by

$$GCDD(\Gamma_G) = \begin{bmatrix} (n-3)(J-I)_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & I_{\frac{n}{2}} & 0_{n/2} \end{bmatrix}$$

Then we obtain $P_{GCDD}(\Gamma_G)(\lambda)$ from the following determinant

$$P_{GCDD}(\Gamma_G)(\lambda) = |\lambda I_{2n-2} - GCDD(\Gamma_G)|$$

$$= \begin{vmatrix} (\lambda+n-3)I_{n-2} - (n-3)J_{n-2} & 0_{(n-2)\times\frac{n}{2}} & 0_{(n-2)\times\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ 0_{\frac{n}{2}\times(n-2)} & \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \end{vmatrix}$$

$$GCDD(\Gamma_G) = \begin{array}{c|cccccccccccc} & a & a^2 & \cdots & a^{\frac{n}{2}-1} & a^{\frac{n}{2}} & a^{\frac{n}{2}+1} & \cdots & a^{n-1} & b & ab & \cdots & a^{n-1}b \\ \hline a & 0 & n-3 & \cdots & n-3 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a^2 & n-3 & 0 & \cdots & n-3 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1} & n-3 & n-3 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ ab & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{\frac{n}{2}-1}b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ a^{\frac{n}{2}}b & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ a^{\frac{n}{2}+1}b & 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a^{n-1}b & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{array} = \begin{bmatrix} 0_{\frac{n}{2}} & I_{\frac{n}{2}} \\ I_{\frac{n}{2}} & 0_{\frac{n}{2}} \end{bmatrix}$$

According to Theorem 2.4 with $M_1 = (\lambda + n - 3)I_{n-2} - (n-3)J_{n-2}$, $M_2 = 0_{((n-2)\times n)}$, $M_3 = 0_{n\times(n-2)}$, and $M_4 = \begin{bmatrix} \lambda I_{\frac{n}{2}} & -I_{\frac{n}{2}} \\ -I_{\frac{n}{2}} & \lambda I_{\frac{n}{2}} \end{bmatrix}$ then we get the form $P_{Z_1(\Gamma_G)}(\lambda) = \begin{vmatrix} M_1 & M_2 \\ 0 & M_4 - M_3M_1^{-1}M_2 \end{vmatrix} = |M_1||M_4 - M_3M_1^{-1}M_2|$ (since $M_3 = 0$). Now we need to obtain $|M_1|$. By Lemma 2.3 with $r = s = t = u = n - 3$, and $n_1 = n_2 = (n-2)/2$, we obtain

$$|M_1| = (\lambda + n - 3)^{n-3}(\lambda - (n-3)^2)$$

Meanwhile for $|M_4|$, Using the same reasoning as in Theorem 3.6's proof for even n, then

$$|M_4| = (\lambda + 1)^{n/2}(\lambda - 1)^{n/2}$$

Therefore,

$$\begin{aligned} P_{GCDD}(\Gamma_G)(\lambda) =& (\lambda + n - 3)^{n-3}(\lambda - (n-3)^2) \\ & (\lambda + 1)^{\frac{n}{2}}(\lambda - 1)^{\frac{n}{2}} \end{aligned}$$

Considering the fact from Theorem 3.7, we can briefly conclude the two following theorems as their implications.

Theorem 3.8. In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, the GCDD-spectral radius of Γ_G is

$$\rho_{GCDD(\Gamma_G)} = \begin{cases} (n-2)^2, & \text{for odd } n \\ (n-3)^2, & \text{for even } n \end{cases}$$

Proof.

1. For the odd n case, based on Theorem 3.3 (1), the characteristic formula of $GCDD(\Gamma_G)$ contains three eigenvalues: $\lambda_1 = -(n-2)$ of multiplicity $n-2$, $\lambda_2 = 0$ of multiplicity n and a single $\lambda_3 = (n-2)^2$. Then, GCDD-spectrum is

   $$\text{Spec}(\Gamma_G) = \{(n-2)^2, 0^n, (-(n-2))^{n-2}\}$$

   Taking the largest of λ_i, for $i = 1, 2, 3$, then we have

   $$\rho_{GCDD(\Gamma_G)} = (n-2)^2$$

2. On the other hand, for even n, Theorem 3.7 (2) shows that $P_{GCDD}(\Gamma_G)(\lambda)$ has four eigenvalues: $\lambda_1 = -(n-3)$ of multiplicity $n-3$, $\lambda_2 = (n-3)^2$ of multiplicity 1, and the last two are $\lambda_{3,4} = \pm 1$, each of multiplicity $n/2$. Here the spectrum of Γ_G is

   $$Spec(\Gamma_G) = \{(n-3)^2, 2^{n/2}, (-2)^{n/2}, (-(n-3))^{n-3}\}$$

   Hence, GCDD-spectral radius for Γ_G is

   $$\rho_{GCDD(\Gamma_G)} = (n-3)^2$$

Theorem 3.9. In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, the GCDD-energy of Γ_G is

$$E_{GCDD}(\Gamma_G) = \begin{cases} 2(n-2)^2, & \text{for odd } n \\ 2(n-3)^2 + n, & \text{for even } n \end{cases}$$

Proof.

1. Taking the eigenvalues from Spec(Γ_G) for odd n, we obtain the greatest common divisor degree energy of Γ_G as follows:

   $$\begin{aligned} E_{GCDD}(\Gamma_G) =& (n-2)|-(n-2)| + (n)|0| \\ & + |(n-2)^2| = 2(n-2)^2 \end{aligned}$$

2. Using Spec(Γ_G) for even n, we get

   $$\begin{aligned} E_{GCDD}(\Gamma_G) =& (n-3)|-(n-3)| + |(n-3)^2| \\ & + \frac{n}{2}|-1| + \frac{n}{2}|1| = 2(n-3)^2 + n \end{aligned}$$

3.3 Discussions

Based on the results from Theorems 3.2, 3.5, 3.6, and 3.9, for odd n, the energy of Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$ is equal to the energy of Γ_G for $G = G_1$, due to one component of Γ_G is isomorphism with the complete graph on $n-1$ vertices, and all other components are isolated vertices. Meanwhile, for even n, the energy of Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$ is the sum of the energy of Γ_G for $G = G_1$ and for $G = G_2$, since Γ_G consists of disconnected two components.

By contrasting the outcomes of Theorems 3.5 and 3.9, we can determine the following:

Corollary 3.10. In Γ_G where $G = G_1 \cup G_2 \subset D_{2n}$,

$$E_{Z_1}(\Gamma_G) = 2 \cdot E_{GCDD}(\Gamma_G).$$

Furthermore, we also obtain the classification of $GCDD$- and Z_1-energies of Γ_G for D_2n, where $G = G_1 \cup G_2 \subset D_{2n}$.

Corollary 3.11. In Γ_G for $G = G_1 \cup G_2 \subset D_{2n}$, Γ_G corresponds with the $GCDD$ and Z_1- matrices are hyperenergetic.

Corollary 3.12. Both $E_{Z_1}(\Gamma_G)$ and $E_{GCDD}(\Gamma_G)$ are never odd numbers, and they also never represent the square root of an odd integer.

The claim in Corollary 3.12 is consistent with the well-known information from Pirzada and Gutman (2008) and Bapat and Pati (2004). Moreover, Romdhini and Nawawi (2022) have formulated the maximum degree ($MaxD$) and minimum degree ($MinD$) energies of Γ_G for D_{2n}, and here we can conclude the relationship as follows:

Corollary 3.11. In Γ_G where $G = G_1 \cup G_2 \subset D_{2n}$,

$$E_{GCDD}(\Gamma_G) = E_{MaxD}(\Gamma_G) = E_{MinD}(\Gamma_G).$$

4. CONCLUSION

The energy of the commuting graph for D_{2n} associated with Z_1 and $GCDD$-matrices have been investigated in this study. We have verified the relationship between both energies, as well as between both energies and previous literature. The obtained energies are identified as hyperenergetic. Moving forward, our future research will identify the graph energy where the vertex set is defined on the ring, such as the prime ideal graph, and extend the graph matrices, for instance, transmission-based matrices.

5. ACKNOWLEDGMENT

The authors would like to thank the referees for their helpful comments or recommendations on this article. Additionally, we would like to thank Universitas Mataram in Indonesia for its partial funding assistance.

REFERENCES

Aschbacher, M. (2000). *Finite Group Theory*. Cambridge University Press, Cambridge

Bapat, R. and S. Pati (2004). Energy of a Graph Is Never an Odd Integer. *Bulletin of Kerala Mathematics Association*, **1**; 129–132

Bhat, K. and S. Shetty (2024). Energy and Spectra of Zagreb Matrix of K-Half Graph. *Engineering Letters*, **32**(4); 736–742

Brauer, R. and K. Fowler (1955). On Groups of Even Order. *Annals of Mathematics*, **62**(3); 565–583

Chattopadhyay, S., P. Panigrahi, and F. Atik (2018). Spectral Radius of Power Graphs on Certain Finite Groups. *Indagationes Mathematicae-New Ser*, **29**; 730–737

Das, K., I. Gutman, I. Milovanovic, E. Milovanovic, and B. Furtula (2018). Degree-Based Energies of Graphs. *Linear Algebra and Its Applications*, **554**; 185–204

Filipovski, S. and R. Jajcay (2021). Bounds for the Energy of Graphs. *Mathematics*, **9**(1687); 1–10

Ganie, H. and Y. Shang (2022). On the Spectral Radius and Energy of Signless Laplacian Matrix of Digraphs. *Heliyon*, **8**; 1–6

Gantmacher, F. (1959). *The Theory of Matrices*, volume 1. Chelsea Publishing Company, New York

Gutman, I. (1978). The Energy of Graph. *Ber. Math. Statist. Sekt. Forschungszenturm Graz.*, **103**; 1–2

Hameed, A., Z. Khan, and M. Tyaglov (2022). Laplacian Energy and First Zagreb Index of Laplacian Integral Graphs. *An. St. Univ. Ovidius Constanta*, **30**(2); 133–160

Horn, R. and C. Johnson (1985). *Matrix Analysis*. Cambridge University Press, Cambridge

Jahanbani, A., R. Khoeilar, and H. Shooshtari (2022). On the Zagreb Matrix and Zagreb Energy. *Asian-European Journal of Mathematics*, **15**(1); 2250019

Li, X., Y. Shi, and I. Gutman (2012). *Graph Energy*. Springer, New York

Pirzada, S. and I. Gutman (2008). Energy of a Graph Is Never the Square Root of an Odd Integer. *Applicable Analysis and Discrete Mathematics*, **2**; 118–121

Rad, N., A. Jahanbani, and I. Gutman (2018). Zagreb Energy and Zagreb Estrada Index of Graphs. *MATCH Communications in Mathematical and in Computer Chemistry*, **79**(2); 371–386

Ramane, H. and S. Shinde (2017). Degree Exponent Polynomial of Graphs Obtained by Some Graph Operations. *Electronic Notes in Discrete Mathematics*, **63**; 161–168

Ramkumar, R. and K. Nagarajan (2017). Greatest Common Divisor Degree Energy of Graphs. *International Journal of Applied and Computational Mathematics*, **11**; 163–171

Ramkumar, R. and K. Nagarajan (2018). Bounds on Greatest Common Divisor Degree Spectral Radius and Greatest Common Divisor Degree Energy of Graphs. *Journal of Applied Science and Computations*, **5**(11); 1348–1353

Romdhini, M. and A. Nawawi (2022). Maximum and Minimum Degree Energy of Commuting Graph for Dihedral Groups. *Sains Malaysiana*, **51**(12); 4145–4151

Romdhini, M., A. Nawawi, and C. Chen (2022). Degree Exponent Sum Energy of Commuting Graph for Dihedral Groups. *Malaysian Journal of Science*, **41**(sp1); 40–46

Shao, Y., Y. Gao, W. Gao, and X. Zhao (2021). Degree-Based Energies of Trees. *Linear Algebra and Its Applications*, **621**; 18–28

Sharmila, D., S. Sujhita, and M. Jebitha (2023). First Zagreb Matrix and Energy of a T_2 Hypergraph. *South East Asian Journal of Mathematics and Mathematical Sciences*, **19**(2); 231–240

Shetty, S. and K. Bhat (2024). Energy and Spectral Radius of Zagreb Matrix of Graph with Applications. *Nanosystems Physics Chemistry Mathematics*, **15**(3); 315–324