Science and Technology Indonesia
e-ISSN:2580-4391 *p*-ISSN:2580-4405
Vol. 3, No. 4, October 2018

Research Paper

On The Infinitely Divisible of Meixner Distribution

Dodi Devianto[1]*, Rahma Diana Safitri[1], Jayanti Herli[1], Maiyastri[1]

[1]*Department of Mathematics, Faculty of Mathematics and Natural Sciences, Andalas Univerrsity Padang 25163, West Sumatra Province, Indonesia*
***Corresponding author**: ddevianto@fmipa.unand.ac.id

Abstract
The log-returns of most financial data show a significant leptokurtosis. For the better fit we showed a special levy process which is called the Meixner process. The Meixner distribution belongs to the class of infinitely divisible distribution characterized by using characteristic function satisfied continuity and complex conjugate properties, and it was proposed as a model for represented efficiently of the log-returns of financial data. The perfect fit of underlying Meixner distribution performing by using *goodness of fit test*.

Keywords
Meixner distribution, infinitely divisible distribution, characteristic function, *goodness of fit test*

Received: 10 May 2018, Accepted: 16 October 2018
https://doi.org/10.26554/sti.2018.3.4.147-150

1. INTRODUCTION

The log-returns of most financial data have an actual kurtosis. The normal distribution has lower kurtosis than log-returns of most financial data, which show a significant leptokurtosis. Therefore, we propose another model which is reached the higher kurtosis in financial data. The model is the special of levy process, that is Meixner process, which was introduce in Grigelionis (1999). The Meixner distribution is a distribution with four parameters there are (α,β,μ,δ)it was introduced in Schouten (2001, 2003). This paper will determine the infinitely divisible of Meixner distribution and it is shown the goodness of fit of Meixner distribution on financial data.

The simulation and estimation of the Meixner distribution was introduced in Grigoletto and Provasi (2009). Kawai (2012) has studied likelihood ratio gradient estimation for Meixner distribution and Levy processes, while Kawai and Masuda (2011) have studied local asymptotic normality property for Meixner Levy processes with discrete observations. Fengler and Melnikov (2017) have introduced the models option pricing GARCH with innovation Meixner. Hass (2018) has investigated the pricing of cliquet options with Meixner processes. Most of the analysis of financial data by previous researcher showed interest on the dynamics of exchange rate. On this paper we will present the analyze of IDR-USD rates and fit the data to Meixner distribution.

This paper will give the property of infinitely divisible of Meixner distribution. In section 2, we introduce the Meixner distribution and the infinitely divisible characteristic function of Meixner distribution. In section 3 we fit the Meixner distribution to our data set of exchange of IDR-USD rates and we perform the result by Histogram and QQ-plot with *p-value* of Kolmogorov-Smirnov test using *WolframMathematica*

2. THE MEIXNER DISTRIBUTION

This section will give definition of Meixner distribution and some properties of characteristic function including its inifinite divisibility.

Definition 1 The probability density function (pdf) of the Meixner distribution with parameter $\theta = (\alpha, \beta, \delta, \mu)$ with notation $X \sim MXR(\alpha, \beta, \delta, \mu)$ that is

$$f_{MXR}(x;\theta) = \frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{\beta(x-\mu)}{\alpha}\right)\left|\Gamma\left(\delta + \frac{i(x-\mu)}{\alpha}\right)\right|^2 \quad (1)$$

where

$$-\infty < x < \infty,\ -\infty < \mu < +\infty,\ \alpha > 0,\ -\pi < \beta < \pi,\ \delta > 0$$

and

$$\left|\Gamma\left(\delta + \frac{i(x-\mu)}{\alpha}\right)\right|^2 = \left(\int_0^\infty \cos\left(\frac{x-\mu}{\alpha}\log y\right) y^{\delta-1}e^{-y}dy\right)^2 + \left(\int_0^\infty \sin\left(\frac{x-\mu}{\alpha}\log y\right) y^{\delta-1}e^{-y}dy\right)^2 \quad (2)$$

The parameters α and β describe the steepness and symmetry, and the parameters δ and μ determine the scale and location of the density function.

The characteristic function φ of a distribution from random variable X, is the Fourier-Stieljes transform of random variable X with distribution function$F(x) = P(X \leq x)$ that is defined in Lukacs (1992) as follows

$$\varphi_X(t) = E(\exp(itX)) = \int_{-\infty}^{+\infty} \exp(itx)dF(x). \quad (3)$$

The characteristic function of the Meixner distribution with parameters $(\alpha, \beta, \delta, \mu)$ is given by

$$\varphi_X(t) = \left(\frac{\cos(\frac{\beta}{2})}{\cosh(\frac{(\alpha t - i\beta)}{2})}\right)^{2\delta} \exp(it\mu) \quad (4)$$

The followings will give some properties of characteristic function from Meixner distribution as theory on supporting its infinite divisibility.

Preposition 1 Let $\varphi_X(t)$ is the characteristic function of Meixner distribution, then $\varphi_X(0)$= 1.

Proof. It is easily to have $\varphi_X(0)$= 1 □

Preposition 2 Let X be random variable of Meixner distribution with characteristic function $\varphi_X(t)$, then characteristic function of $-X$ is conjugate from characteristic function $\varphi_X(t)$ that is $\overline{\varphi_X(t)}$.

Proof. Let $Y = -X$ then we get $\frac{dy}{dx} = -1$, so that we have

$$|J| = \left|\frac{dy}{dx}\right| = |-1| = 1.$$

Since $Y = -X$, the characteristic function of random variable $-X$ is

$$\begin{aligned}
\varphi_{-X}(t) &= \int_{-\infty}^{\infty} \exp\ it(-x)f(-x)dx \\
&= \int_{-\infty}^{\infty} \exp\ it(-x)\frac{(2\cos(\beta/2))^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \exp\left(\frac{\beta(x-\mu)}{\alpha}\right) \\
&\quad \left|\Gamma\left(\delta + \frac{i(x-\mu)}{\alpha}\right)\right|^2 dx \\
&= \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\beta - i\alpha t}{2}\right)}\right)^{2\delta} \exp(-it\mu) \int_{-\infty}^{\infty} \frac{\left(2\cos\left(\frac{\beta - i\alpha t}{2}\right)\right)^{2\delta}}{2\alpha\pi\Gamma(2\delta)} \\
&\quad \exp\left(\frac{\beta - i\alpha t}{\alpha}(-x-\mu)\right)\left|\Gamma\left(\delta + \frac{i(-x-\mu)}{\alpha}\right)\right|^2 dx \\
&= \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\beta - i\alpha t}{2}\right)}\right)^{2\delta} \exp(-it\mu) \\
&\quad \int_{-\infty}^{\infty} f_{MXR}(-x; \alpha, (\beta - i\alpha t), \delta, \mu)\, dx \\
&= \left(\frac{cos\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\beta - i\alpha t}{2}\right)}\right)^{2\delta} \exp(-it\mu) \\
&= \overline{\varphi_X(t)}. \quad (5)
\end{aligned}$$

□

Preposition 3 Characteristic function of Meixner distribution is continuous.

Proof. Let $\varphi_X(t)$ is characteristic function of Meixner distribution and let $h = s - t$ for $s > t$. It will shows that for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$, therefore $|\varphi_X(s) - \varphi_X(t)| < \varepsilon$, then $|s - t| < \delta(\varepsilon)$. We have the following equation

$$\begin{aligned}
|\varphi_X(s) - \varphi_X(t)| &= \left|\left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\alpha s - i\beta}{2}\right)}\right)^{2\delta} \exp(is\mu) - \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\alpha t - i\beta}{2}\right)}\right)^{2\delta} \exp(it\mu)\right| \\
&= \left|\left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\alpha(h+t) - i\beta}{2}\right)}\right)^{2\delta} \exp(i(h+t)\mu) - \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cos\left(\frac{\alpha t - i\beta}{2}\right)}\right)^{2\delta} \exp(it\mu)\right|
\end{aligned}$$

For $h = s - t$, then $h \to 0$ and $|\varphi_X(s) - \varphi_X(t)| \to 0$ and then for every $\varepsilon > 0$ there exists $\delta(\varepsilon)$ in form $\delta(\varepsilon) < \varepsilon$, then $|\varphi_X(s) - \varphi_X(t)| < \varepsilon$, then $|s - t| < \delta(\varepsilon)$. □

Lemma 1. *Let X be random variable of Meixner, X∼MXR(α, β, δ, μ), then the expected value, variance, skewness and kurtosis of random variable X are given by*

$$E(X) = \alpha\delta\tan\left(\frac{\beta}{2}\right) + \mu$$

$$VAR(X) = \frac{\alpha^2\delta}{2}\sec^2\left(\frac{\beta}{2}\right)$$

$$SKEW(X) = \sqrt{\frac{2}{\delta}} sin\left(\frac{\beta}{2}\right)$$

$$KURT(X) = 3 + \frac{2 - cos(\beta)}{\delta}$$

Proof. Let $\kappa_X{}^{(r)}(t)$ denoted the r^{th} derivative of $\kappa_X(t)$ of the cumulant generating function of the Meixner distribution with respect to t. then it can be shown that:

$$\kappa_X(t) = 2\delta\ \ln\left(\cos\left(\frac{\beta}{2}\right)\right) - 2\delta\ \ln\left(\cos\left(\frac{\alpha t + \beta}{2}\right)\right) + \mu t$$

$$\begin{aligned}\kappa_X{}^{(1)}(t) &= 0 - 2\delta \frac{1}{\left(\cos\left(\frac{\alpha t+\beta}{2}\right)\right)}\left(-\sin\left(\frac{\alpha t+\beta}{2}\right)\right)\left(\frac{\alpha}{2}\right) + \mu \\ &= \alpha\delta\left(\frac{\sin\left(\frac{\alpha t+\beta}{2}\right)}{\cos\left(\frac{\alpha t+\beta}{2}\right)}\right) + \mu \\ &= \alpha\delta \; tan\left(\frac{\alpha t+\beta}{2}\right) + \mu\end{aligned}$$

$$\kappa_X{}^{(2)}(t) = \frac{\alpha^2\delta}{2}\; sec^2\left(\frac{\alpha t+\beta}{2}\right)$$

$$\kappa_X{}^{(3)}(t) = \frac{\alpha^3\delta}{2}\; sec^2\left(\frac{\alpha t+\beta}{2}\right)\; tan\left(\frac{\alpha t+\beta}{2}\right)$$

$$\kappa_X{}^{(4)}(t) = \frac{\alpha^4\delta}{2}\; sec^2\left(\frac{\alpha t+\beta}{2}\right)\; tan^2\left(\frac{\alpha t+\beta}{2}\right) + \frac{\alpha^4\delta}{4}\; sec^4\left(\frac{\alpha t+\beta}{2}\right)$$

We can make moments of meixner distribution be able by derivative of the cumulant generating function, so that we have

Mean	$\bar{x} = E[X] = \kappa_X{}^{(1)}(0) = \alpha\delta\tan\left(\frac{\beta}{2}\right) + \mu$
Varians	$s^2 = Var[X] = \kappa_X{}^{(2)}(0) = \frac{\alpha^2\delta}{2}\sec^2\left(\frac{\beta}{2}\right)$
Skewness	$b_1 = Skew[X] = \frac{\kappa_X{}^{(3)}(0)}{(\kappa_X{}^{(2)}(0))^{\frac{3}{2}}} = \sqrt{\frac{2}{\delta}}\sin\left(\frac{\beta}{2}\right)$
Kurtosis	$b_2 = Kurt[X] = 3 + \frac{\kappa_X{}^{(4)}(0)}{(\kappa_X{}^{(2)}(0))^2} = 3 + \frac{2-\cos(\beta)}{\delta}$

□

The convolution of random variables from a Meixner distribution as the main result is established in this paper. This convolution as the sum of independent and identically random variable from a Meixner distribution is stated in the following way. Let X_i be random variables from Meixner distribution, where $X_i \sim MXR(\alpha, \beta, \delta, \mu)$. The convolution of Meixner distribution is defined as random variable $S_n = \sum_{i=1}^n X_i$.
The characteristic function of S_n is given as follows

$$\begin{aligned}\varphi_{S_n}(t) &= \sum_{i=1}^{n}\varphi_{X_i}(t) \\ &= \varphi_{X_1} + \varphi_{X_2} + \ldots + \varphi_{X_n} \\ &= \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha t - i\beta)}{2}\right)}\right)^{2\delta} e^{it\mu} + \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha t - i\beta)}{2}\right)}\right)^{2\delta} e^{it\mu} \\ &\quad + \ldots + \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha t - i\beta)}{2}\right)} e^{it\mu}\right)^{2\delta} e^{it\mu} \\ &= \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha t - i\beta)}{2}\right)}\right)^{2n\delta} e^{itn\mu} \qquad (6)\end{aligned}$$

This is show that S_n also has Meixner distribution with parameter $(\alpha, \beta, n\delta, n\mu)$.

3. THE INFINITELY DIVISIBLE OF MEIXNER DISTRIBUTION

The infinitely divisible of some distribution function is defined if for every n positive integer there is a distribution function F_n such as the distribution F is convolution n time's of F_n, where $F = (F_n * F_n * \ldots * F_n)$, see Chung (2001). An equivalent definition of infinitely divisible distribution is also given on property of characteristic function. Steutel and Harn (2004) give its undertanding, suppose $\varphi_X(t)$ is the characteristic function of the distribution F. If for every positive integer n, there exists a function φ_{X_n} as the n-th power of a characteristic function becomes the characteristic function $\varphi_X(t)$, we say that the distribution is infinitely divisible, then $\varphi_X(t) = \left(\varphi_{X_n}(t)\right)^n$
Preposition 4 Meixner distribution is infinitely divisible.

Proof. There is exist a characteristic function of Meixner distribution with

$$\varphi(t) = \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha t - i\beta)}{2}\right)}\right)^{\frac{2\delta}{n}} \exp\left(\frac{(it\mu)}{n}\right) \qquad (7)$$

as the characteristic function of Meixner distribution with parameters $\left(\alpha, \beta, \frac{\delta}{n}, \frac{\mu}{n}\right)$. Then we have the following relation

$$\begin{aligned}\left(\varphi_n\left(t; \alpha, \beta, \frac{\delta}{n}, \frac{\mu}{n}\right)\right)^n &= \left(\left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha t - i\beta)}{2}\right)}\right)^{\frac{2\delta}{n}} \exp\left(\frac{it\mu}{n}\right)\right)^n \\ &= \left(\frac{\cos\left(\frac{\beta}{2}\right)}{\cosh\left(\frac{(\alpha t - i\beta)}{2}\right)}\right)^{2\delta} \exp\left(it\mu\right) \\ &= \varphi(t; \alpha, \beta, \delta, \mu). \qquad (8)\end{aligned}$$

A consequence of infinitely divisible is that the characteristic function of the Meixner distribution is infinitely divisible, since we have $\varphi_X(t; \alpha, \beta, \delta, \mu) = \left[\varphi_{X(n)}\left(t; \alpha, \beta, \frac{\delta}{n}, \frac{\mu}{n}\right)\right]^n$. □

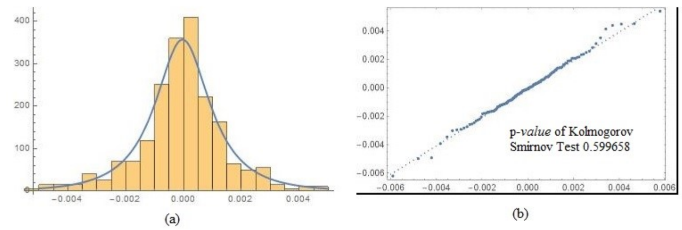

Figure 1. Histogram and QQ-plot of goodness of fit performance from Meixner distribution

4. THE GOODNESS OF FIT OF MEIXNER DISTRIBUTION

In this paper we studied the dynamics data in exchange of IDR-USD rates as the application of the infinitely divisible Meixner distribution. The period of data analysis of exchange rate IDR (Indonesian Rupiah) to USD (Dolar of American) is from Januari 4^{th} 2016 until Augustus 31^{th} 2017, totally of data are 407. Exchange rate IDR-USD we get from BI (Bank Indonesia). The goodness of fit performance shows graphical by histogram, QQ-plot and also with statistical test by Kolmogorov-Smirnov. The QQ-plot is a graphical method that plots the empirical quantiles of the fitted distribution. And the numerical of *p-value* from Kolmogorov-Smirnov test is must been higher than 0.05. The histogram and QQ-plot of the empirical of the Meixner distribution fitted the log-returns of data was showed in (a). The QQ-plot was showed in (b), it shows that the points lie on or very near to the straight line and it confirm the Meixner distribution is the better fit for analyzing financial data. The Quantile-plot from Meixner distribution in log return of exchange rate in Figure (1.b), with the *p-value* of Kolmogorov-Smirnov test. 0.599658 and it is greater than 0.05, then we conclude that the data is distributed to Meixner distribution.

The consequence of infinitely divisble Meixner distribution, the we can divide the data into 12, it is to show the change of in log return of exchange rate hourly. The originnal curve of probability density function of Meixner distribution is in blue line and the infinitely divisible of Meixner distribution (log return of exchange rate hourly) in red line. Parameters of exchange rate IDR-USD (daily data) is $\alpha = 0.00439032$, $\beta = -0.0125685$, $\mu = -0.0000354$, $\delta = 0.2713906$. And then we make the data exchange rate IDR-USD to hourly, then we have $\alpha = 0.00439032$, $\beta = -0.0125685$, $\mu = -0.000001475$, $\delta = 0.01130794$ this performance shows that the Meixner distribution is infinitely divisible such as in Theorem 3.1.

5. CONCLUSION

The infinitely divisible Meixner distribution is determined by using the property of characteristic function. The characteristic

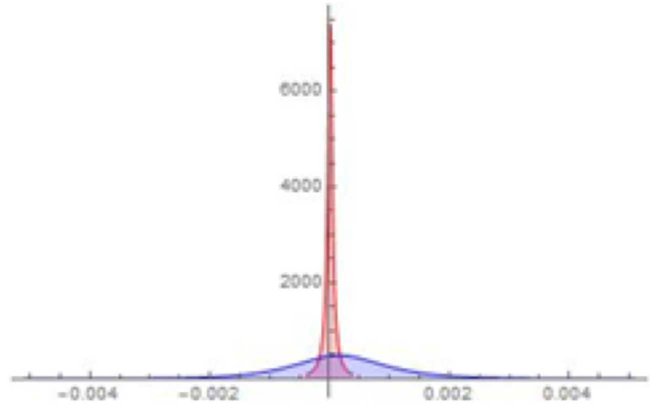

Figure 2. Probability density function of Meixner distribution for daily (blue line) and hourly (red line) from log return of exchange rates IDR-USD

function which is defined as Fourier-Stieltjes transform has performed the continuity property. The financial data of log return exchange rate IDR-USD has shown a significant where Meixner distribution has better fit as special Levy process.

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