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Research Paper

Incorporating $\breve{q}$-Rung Picture Fuzzy Frank Prioritized Weighted Aggregators with Multimoora Strategy for Decision Making

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Abstract
The Frank norm operators and $\breve{q}$-rung picture fuzzy sets offer a versatile framework due to their adjustable parameters. Conversely, prioritized weighted operators are crucial for indicating the relative significance of alternatives and attributes. In this research, we have devised Frank's prioritized weighted procedure for $\breve{q}$-rung picture fuzzy numbers. Thus, the $\breve{q}$-rung picture fuzzy prioritized weighted Frank averaging ($\breve{q}$-RPFPWFA) and geometric ($\breve{q}$-RPFPWFG) operators are presented and analyzed along with some of their properties. Utilizing these proposed operators, a novel approach is established, combining the enhanced MULTIMOORA (MM) procedure with unspecified weight data. A numerical example regarding the multiple attribute group decision-making (MAGDM) problem of selecting the financial director of a residential society is handled based upon the aforementioned procedure. Furthermore, we have compared the results with those obtained using available operators. Evidently, the novel $\breve{q}$-RPFPWFA and $\breve{q}$-RPFPWFG aggregators with the MM technique yield reasonable and consistent outcomes for solving MAGDM.

Keywords
Frank Norm, $\breve{q}$-Rung Picture Fuzzy, Prioritized Weighted, Multimoora, Group Decision-Making, Aggregators

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1. INTRODUCTION

Multi-criteria decision-making (MCDM) stands as a notable subfield within the realm of decision science, that has garnered significant attention from researchers. In this procedure the most precise solution is determined through the evaluation of alternatives against multiple criteria. Owing to the inaccuracy and vagueness in numerous decision making (DM) issues, Zadeh (1965) proposed a highly valuable tool known as fuzzy set (FS) to handle such issues. Zadeh only discussed an element's membership grade or positive grade (PG) in a FS. Subsequently, Atanassov (1986) extended upon the concept of fuzziness to intuitionistic fuzzy set (IFS) by Putting forth another parameter termed as the non-membership grade or negative grade (NG). Building upon this, (Cuong, 2014) presented the picture fuzzy set (PFS), which further extended IFS by including a neutral grade or indeterminacy grade (IG). Thus PFSs have garnered growing interest for addressing MCDM issues (Peng et al., 2020; Wang et al., 2018; Wei, 2017). Ashraf and Abdullah (2019) employed spherical fuzzy operators in MCDM issue.

Pythagorean fuzzy sets (PyFSs) were initially devised by Yager (2013) which encompass a broader structure than IFS. But in the cases, where PG is 0.9 and NG is 0.5, IFS and PyFS are not suitable. Thus, Yager (2017) introduced $\breve{q}$-rung orthopair fuzzy numbers ($\breve{q}$-ROFNs) subject to the condition that sum of $\breve{q}^{th}$ power of PG and NG should not exceed one where as Liu and Wang (2018) presented its operating rules. On the other hand, Li et al. (2018) elucidated the idea of $\breve{q}$-rung picture fuzzy set ($\breve{q}$-RPFS) with the parameter $\breve{q}$ provides a better means of expressing expert opinions without any restrictions in terms of PG, IG and NG. Similarly $\breve{q}$-RPFSs have gained focus for addressing MCDM issues (Akram et al., 2022; He et al., 2019; Liu et al., 2020).

However, Frank (1979) developed a category of triangular norm and conorm known as the Frank t-norm and t-conorm (FTT). Consequently, various studies have proposed accumulation operators that utilize FTT to incorporate ambiguous data into different fuzzy situations. For example, Zhang et al. (2015) applied it in the framework of IFS, while Seikh and Mandal (2021, 2022) it in the context of PF and $\breve{q}$-ROF settings. Chitra and Prabakaran (2023) also employed Frank aggregation operators (aggregators) in the context of $\breve{q}$-RPFS. However, Hasnan et al. (2024) presented weighted geometric Bonferroni aggregators in the triangular fuzzy framework.

The initial notions regard to FSs and their extensions were based on the assumption that both criteria and decision makers held the same priority level. Yet, in addressing the prioritization issue, analysts introduced specific prioritized aggregators (PAs) in various scenarios, recognizing that different criteria might have varying levels of importance. For instance, Yager (2008) took an initiative by establishing the notion of PAs. Arora and Garg (2019) formulated a group decision-making procedure centred on linguistic PAs and examined its core characteristics. Garg and Nancy (2018) devised a MCDM approach in accordance with the prioritized Muirhead mean aggregators under the neutrosophic set domain. Akram et al. (2020) formulated a group decision-making procedure based upon spherical fuzzy PAs and examined its core characteristics.

MOORA is a multi-objective optimization method initially created by Brauers and Kazimieras Zavadskas (2006). Later, they introduced the MOORA procedure along with the complete multiplicative aspect to MOORA, resulting in the emergence of MULTIMOORA (Brauers and Zavadskas, 2010). Many researchers enhanced MM procedure in the context of various fuzzy sets to handle the multi-attribute group decision-making (MAGDM) issues. Tian et al. (2022) enhanced MM PF based upon Schweizer Sklar prioritized aggregators; Riaz et al. (2022) enhanced MM $\check{q}$-ROF procedure with Einstein aggregators. Aydemir and Yilmaz Gündüz (2020) enhanced MM $\check{q}$-ROF based upon Dombi prioritized aggregators.

We will outline the purpose and primary contributions of this article:

- From previous research, it is evident that there has not been an investigation regarding the enhancement of the MULTIMOORA procedure with Frank prioritized weighted aggregators in the context of $\check{q}$-RPF set.
- Thus, motivated by the existing work on prioritized aggregators and flexibility of $\check{q}$-RPFS, we have developed a novel approach incorporating the MULTIMOORA strategy with fuzzy prioritized weighted aggregators, and thus devised the $\check{q}$-rung picture fuzzy prioritized weighted Frank averaging ($\check{q}$-RPFPWFA) and geometric ($\check{q}$-RPFPWFG) aggregators along with their core characteristics.
- To address the MAGDM issue with unspecified weight information, based upon the suggested approach.

The format of this article unfolds as follows: Section 2 provides fundamental definitions that aid in understanding the introduced aggregators. In Section 3, we introduce the prioritized aggregators, specifically the $\check{q}$-RPFPWFA and the $\check{q}$-RPFPWFG operators. This section also elucidates some valuable characteristics of these operators. Next, an enhanced MULTIMOORA approach based upon Frank PAs for unknown weight data has been formulated in the $\check{q}$-RPF context. Moreover, a numerical illustration of the MAGDM issue, based on the proposed approach is provided, accompanied by an examination of the parameters of the aggregators. A comparison of our devised methodology with previously established methodologies in the literature is also discussed in detail. Section 4 presents the concluding remarks, followed by the acknowledgements in Section 5.

2. EXPERIMENTAL SECTION

2.1 Preminalies

To enhance understanding of the fundamental concepts, this section provides a description of $\check{q}$-rung orthopair fuzzy sets and their extension, $\check{q}$-rung picture fuzzy sets. Following that, the comparison of two $\check{q}$-RPF numbers and their operating rules are specified.

Definition 1 Yager (2017) Let Y indicate the universal set. The $\check{q}$-rung orthopair fuzzy set ϱ' on Y is elucidated as: $\varrho' = \{\langle y, \Psi_{\varrho'}(y), \Phi_{\varrho'}(y)\rangle : y \in Y\}$, where the PG of ϱ' expressed by $\Psi_{\varrho'}(y)$ and the NG of ϱ' expressed as $\Phi_{\varrho'}(y)$ in a way that meets the requirement, $0 \le (\Psi_{\varrho'}(y))^{\check{q}} + (\Phi_{\varrho'}(y))^{\check{q}} \le 1$, $\check{q}$ is a positive integer. Also, the indeterminacy grade of y in ϱ' is obtained by, $\left(1 - (\Psi_{\varrho'}(y))^{\check{q}} - (\Phi_{\varrho'}(y))^{\check{q}}\right)^{(1/\check{q})}$. For ease, a $\check{q}$-rung orthopair fuzzy number is indicated as $\varrho'_j = \left\langle \Psi_{\varrho'_j}, \Phi_{\varrho'_j} \right\rangle$.

Definition 2 Li et al. (2018) Let Y indicate the universal set. The $\check{q}$-rung picture fuzzy set ϱ on Y is elucidated as: $\varrho = \{\langle y, \Psi_{\varrho}(y), \Theta_{\varrho}(y), \Phi_{\varrho}(y)\rangle : y \in Y\}$, where the PG of ϱ stated as $\Psi_{\varrho}(y)$, the IG of ϱ stated as $\Theta_{\varrho}(y)$, and the NG of ϱ stated as $\Phi_{\varrho}(y)$ in a way that meets the requirement, $0 \le (\Psi_{\varrho}(y))^{q} + (\Theta_{\varrho}(y))^{\check{q}} + (\Phi_{\varrho}(y))^{\check{q}} \le 1$, $\check{q}$ is a positive integer. Also, the refusal grade of y in ϱ is obtained by, $\left(1 - (\Psi_{\varrho}(y))^{\check{q}} - (\Theta_{\varrho}(y))^{\check{q}} - (\Phi_{\varrho}(y))^{\check{q}}\right)^{(1/\check{q})}$. For ease, a $\check{q}$-rung picture fuzzy number is indicated as $\varrho_j = \langle \Psi_{\varrho_j}, \Theta_{\varrho_j}, \Phi_{\varrho_j}\rangle$.

Definition 3 Akram et al. (2020) Consider $\varrho = (\Psi_{\varrho}, \Theta_{\varrho}, \Phi_{\varrho})$ as a $\check{q}$-rung picture fuzzy number. The score metric (function) S_{ϱ}, is established in the following manner:

$$S_{\varrho} = \frac{\left(2 + \Psi_{\varrho}^2 - \Theta_{\varrho}^2 - \Phi_{\varrho}^2\right)}{3} \tag{1}$$

Let $\varrho_1 = (\Psi_{\varrho_1}, \Theta_{\varrho_1}, \Phi_{\varrho_1})$ and $\varrho_2 = (\Psi_{\varrho_2}, \Theta_{\varrho_2}, \Phi_{\varrho_2})$ be two $\check{q}$-RPFNs. To compare ϱ_1 and ϱ_2,

i If $S_{\varrho_1} > S_{\varrho_2}$ then $\varrho_1 > \varrho_2$.

ii If $S_{\varrho_2} > S_{\varrho_1}$ then $\varrho_2 > \varrho_1$.

Definition 4 Liu and Wang (2018) The operating principles of $\check{q}$-RPFNs are given below:

i $\varrho_1 \bigoplus \varrho_2 = \left\langle \left(\Psi_{\varrho_1}^{\check{q}} + \Psi_{\varrho_2}^{\check{q}} - \Psi_{\varrho_1}^{\check{q}}\Psi_{\varrho_2}^{\check{q}}\right)^{\frac{1}{\check{q}}}, \Theta_{\varrho_1}\Theta_{\varrho_2}, \Phi_{\varrho_1}\Phi_{\varrho_2} \right\rangle$.

ii $\varrho_1 \bigotimes \varrho_2 = \left\langle \Psi_{\varrho_1}\Psi_{\varrho_2}, \left(\Theta_{\varrho_1}^{\check{q}} + \Theta_{\varrho_2}^{\check{q}} - \Theta_{\varrho_1}^{\check{q}}\Theta_{\varrho_2}^{\check{q}}\right)^{\frac{1}{\check{q}}}, \left(\Phi_{\varrho_1}^{\check{q}} + \Phi_{\varrho_2}^{\check{q}} - \Phi_{\varrho_1}^{\check{q}}\Phi_{\varrho_2}^{\check{q}}\right)^{\frac{1}{\check{q}}} \right\rangle$

iii $\check{\lambda}\varrho = \left\langle \left(1 - \left(1 - \Psi_{\varrho}^{\check{q}}\right)^{\check{\lambda}}\right)^{\frac{1}{\check{q}}}, \Theta_{\varrho}^{\check{\lambda}}, \Phi_{\varrho}^{\check{\lambda}} \right\rangle, \check{\lambda} > 0$.

iv $\varrho^{\check{\lambda}} = \left\langle \Psi_{\varrho}^{\check{\lambda}}, \left(1 - \left(1 - \Theta_{\varrho}^{\check{q}}\right)^{\check{\lambda}}\right)^{\frac{1}{\check{q}}}, \left(1 - \left(1 - \Phi_{\varrho}^{\check{q}}\right)^{\check{\lambda}}\right)^{\frac{1}{\check{q}}} \right\rangle, \check{\lambda} > 0$.

Definition 5 Frank (1979) If we consider real numbers u and v within the closed unit interval $[0, 1]$ then the Frank t-norm and Frank t-conorm are determined as follows:

$F(u,v) = \log_{\breve{\gamma}}\left(1 + \frac{(\breve{\gamma}^{u}-1)(\breve{\gamma}^{v}-1)}{\breve{\gamma}-1}\right)$.
$F'(u,v) = 1 - \log_{\breve{\gamma}}\left(1 + \frac{(\breve{\gamma}^{1-u}-1)(\breve{\gamma}^{1-v}-1)}{\breve{\gamma}-1}\right)$.
where $(u,v) \in [0,1] \times [0,1]$ and $\breve{\gamma} \neq 1$

Chitra and Prabakaran (2023) has presented and developed $\breve{q}$-rung picture fuzzy averaging and geometric aggregators based on the Frank norm operations for $\breve{q}$-rung picture fuzzy numbers, which were discussed already. Initially Yager (2008), presented the prioritized weighted arithmetic (PWA) operator, is described as follows:

Definition 6 Consider a set of criteria, $R = R_1, R_2, \ldots, R_p$. These criteria are prioritized based on a linear ordering, where R_1 is considered to have higher priority than R_2, R_2 has higher priority than R_3, and so forth, with the relationship expressed as $R_1 > R_2 > R_3 > \ldots > R_p$. This means that, criteria R_l is given greater importance than R_m if the subscript l is smaller than m. The value $R_l\ (g)$ represents the performance of any alternative g when assessed against the specific criterion R_l and the performance value $R_l\ (g)$ falls within the range $[0,1]$. Suppose,

$$PWA(R_l(g)) = \sum_{l=1}^{p} \breve{\omega}_l R_l(g) \tag{2}$$

where $\breve{\omega}_l = \frac{H_l}{\sum_{l=1}^{p} H_l}$ and $H_l = \prod_{m=1}^{l-1} R_m(g)(l = 2, \ldots, p)$, $H_1 = 1$. It is called as prioritized weighted arithmetic operator.

3. RESULTS AND DISCUSSION

3.1 $\breve{q}$-Rung Picture Fuzzy Prioritized Weighted Frank Aggregators

Within this part, we have presented two types of aggregators: $\breve{q}$-rung picture fuzzy prioritized weighted Frank arithmetic ($\breve{q}$-RPFPWFA) and $\breve{q}$-rung picture fuzzy prioritized weighted Frank geometric ($\breve{q}$-RPFPWFG) operators. These operators are discussed in the context of unspecified weight information. The core features of $\breve{q}$-RPFPWFA and $\breve{q}$-RPFPWFG aggregators are then explored.

3.1.1 $\breve{q}$-Rung Picture Fuzzy Prioritized Weighted Frank Arithmetic ($\breve{q}$-RPFPWFA) Aggregator

In this segment, $\breve{q}$-rung picture fuzzy prioritized weighted Frank arithmetic ($\breve{q}$RPFPWFA) aggregator is proposed and its characteristics are proved.

Definition 7 For a set of $\breve{q}$-RPFNs denoted as ϱ_1, with each ϱ_1 being represented as $\varrho_l = \left\langle \Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l} \right\rangle$ for $l = (1,2,\ldots,p)$ the $\breve{q}$- RPFPWFA aggregator is formally defined and developed based on Equation 2. A mapping is given by $\breve{q}$-RPFPWFA: $\varrho^p \to \varrho$, where

$$\breve{q} - \text{RPFPWFA}(\varrho_1, \varrho_2, \ldots, \varrho_p) = \bigoplus_{l=1}^{p} \left(\frac{C_l}{\sum_{l=1}^{p} C_l} \varrho_l \right)$$
$$\left(\frac{C_1}{\sum_{l=1}^{p} C_l} \varrho_1 \bigoplus \frac{C_2}{\sum_{l=1}^{p} C_l} \varrho_2 \bigoplus \cdots \bigoplus \frac{C_p}{\sum_{l=1}^{p} C_l} \varrho_p \right)$$

Where, $C_l = \prod_{m=1}^{l-1} S(\varrho_m)$, $l = (2,\ldots,p)$ with $C_1 = 1$ and $S(\varrho_m)$ signifies the score associated with ϱ_m and can be determined using Equation (1).

Theorem 1 The $\breve{q}$-RPFPWFA aggregator is used to combine a collection of $\breve{q}$-RPFNs represented by $\varrho_l = \langle \Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l} \rangle$ for $l = (1,2,\ldots,p)$. The combined value of these $\breve{q}$-RPFNs by applying the $\breve{q}$-RPFPWFA aggregator is another $\breve{q}$-RPFN. The $\breve{q}$-RPFPWFA aggregator is provided below:

$$\breve{q} - \text{RPFPWFA}(\varrho_1, \varrho_2, \ldots, \varrho_p) = \bigoplus_{l=1}^{p} \left(\frac{C_l}{\sum_{l=1}^{p} C_l} \varrho_l \right)$$

$$= \left(1 - \log_{\breve{\gamma}} \left(1 + \prod_{l=1}^{p} \left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}} - 1 \right)^{\frac{C_1}{\sum_{l=1}^{p} C_1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}} \left(1 + \prod_{l=1}^{p} \left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}} - 1 \right)^{\frac{C_1}{\sum_{l=1}^{p} C_1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}} \left(1 + \prod_{l=1}^{p} \left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}} - 1 \right)^{\frac{C_1}{\sum_{l=1}^{p} C_1}} \right) \right)^{\frac{1}{\breve{q}}} \tag{3}$$

where $C_1 = \prod_{m=1}^{l-1} S_{(\varrho_m)}$, $l = (2,\ldots,p)$ with $C_1 = 1$ and $S_{(\varrho_m)}$ signifies the score associated with ϱ_m.

Proof. We establish the validity of this theorem through an inductive proof in the following manner:

For $p = 2$, by definition 7, we get

$$\breve{q} - \text{RPFPWFA}(\varrho_1, \varrho_2) = \bigoplus_{l=1}^{2} \frac{C_l}{\sum_{l=1}^{2} C_l} \varrho_l = \frac{C_1}{\sum_{l=1}^{2} C_l} \varrho_1 \bigoplus \frac{C_2}{\sum_{l=1}^{2} C_l} \varrho_2$$

$$= \left\{ \left(1 - \log_{\breve{\gamma}} \left(1 + \frac{(\breve{\gamma}^{1-\Psi^{\breve{q}}\varrho_1} - 1)^{\frac{C_1}{\sum_{l=1}^{2} C_1}}}{(\breve{\gamma}-1)^{\frac{C_1}{\sum_{l=1}^{2} C_1} - 1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}} \left(1 + \frac{(\breve{\gamma}^{\Theta^{\breve{q}}\varrho_1} - 1)^{\frac{C_1}{\sum_{l=1}^{2} C_1}}}{(\breve{\gamma}-1)^{\frac{C_1}{\sum_{l=1}^{2} C_1} - 1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}} \left(1 + \frac{(\breve{\gamma}^{\Phi^{\breve{q}}\varrho_1} - 1)^{\frac{C_1}{\sum_{l=1}^{2} C_1}}}{(\breve{\gamma}-1)^{\frac{C_1}{\sum_{l=1}^{2} C_1} - 1}} \right) \right)^{\frac{1}{\breve{q}}} \right\}$$

$$\bigoplus \left\{ \left(1 - \log_{\breve{\gamma}} \left(1 + \frac{(\breve{\gamma}^{1-\Psi^{\breve{q}}\varrho_2} - 1)^{\frac{C_1}{\sum_{l=1}^{2} C_1}}}{(\breve{\gamma}-1)^{\frac{C_1}{\sum_{l=1}^{2} C_1} - 1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}} \left(1 + \frac{(\breve{\gamma}^{\Theta^{\breve{q}}\varrho_2} - 1)^{\frac{C_1}{\sum_{l=1}^{2} C_1}}}{(\breve{\gamma}-1)^{\frac{C_1}{\sum_{l=1}^{2} C_1} - 1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}} \left(1 + \frac{(\breve{\gamma}^{\Phi^{\breve{q}}\varrho_2} - 1)^{\frac{C_1}{\sum_{l=1}^{2} C_1}}}{(\breve{\gamma}-1)^{\frac{C_1}{\sum_{l=1}^{2} C_1} - 1}} \right) \right)^{\frac{1}{\breve{q}}} \right\}$$

$$= \left(1 - \log_{\breve{\gamma}} \left(1 + \prod_{l=1}^{2} \left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}} - 1 \right)^{\frac{C_1}{\sum_{l=1}^{2} C_1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}} \left(1 + \prod_{l=1}^{2} \left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}} - 1 \right)^{\frac{C_1}{\sum_{l=1}^{p} C_1}} \right) \right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}} \left(1 + \prod_{l=1}^{2} \left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}} - 1 \right)^{\frac{C_1}{\sum_{l=1}^{2} C_1}} \right) \right)^{\frac{1}{\breve{q}}} \quad \text{since} \sum_{l=1}^{2} \frac{C_l}{\sum_{l=1}^{2} C_l} = 1$$

It turns out to be correct for $p = 2$. If the consequence is correct for $p = h$, then

$$=\left(1-\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{h}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{h}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{h}\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{h}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{h}\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{h}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$

Now for $p = h+1$,

$$\breve{q}-\text{RPFPWFA}(\varrho_1,\varrho_2,\ldots,\varrho_h,\varrho_{h+1}) = \bigoplus_{l=1}^{h+1}\frac{C_l}{\sum_{l=1}^{h+1}C_l}\varrho_l$$

$$=\bigoplus_{l=1}^{h}\frac{C_1}{\sum_{l=1}^{h+1}C_l}\varrho_1\bigoplus\frac{C_{h+1}}{\sum_{l=1}^{h+1}C_l}\varrho_{h+1}$$

$$=\left\{\left(1-\log_{\breve{\gamma}}\left(1+\frac{\prod_{l=1}^{h}(\breve{\gamma}^{1-\Psi^{\breve{q}}\varrho_l}-1)^{\frac{C_1}{\sum_{l=1}^{h+1}}}}{(\breve{\gamma}-1)^{\sum_{l=1}^{h}\frac{C_1}{\sum_{l=1}^{h+1}C_1}-1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\frac{\prod_{l=1}^{h}(\breve{\gamma}^{\Theta^{\breve{q}}\varrho_l}-1)^{\frac{C_1}{\sum_{l=1}^{h+1}C_1}}}{(\breve{\gamma}-1)^{\sum_{l=1}^{h}\frac{C_1}{\sum_{l=1}^{h+1}C_1}-1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\frac{(\breve{\gamma}^{\Phi^{\breve{q}}\varrho_l}-1)^{\frac{C_1}{\sum_{l=1}^{h+1}C_1}}}{(\breve{\gamma}-1)^{\sum_{l=1}^{h}\frac{C_1}{\sum_{l=1}^{h+1}C_1}-1}}\right)\right)^{\frac{1}{\breve{q}}}\right\}\bigoplus\left\{\left(1-\log_{\breve{\gamma}}\left(1+\frac{(\breve{\gamma}^{1-\Psi^{\breve{q}}\varrho_{h+1}}-1)^{\frac{C_{h+1}}{\sum_{l=1}^{h+1}C_1}}}{(\breve{\gamma}-1)^{\frac{C_{h+1}}{\sum_{l=1}^{h+1}C_1}-1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\frac{(\breve{\gamma}^{\Theta^{\breve{q}}\varrho_{h+1}}-1)^{\frac{C_{h+1}}{\sum_{l=1}^{h+1}C_1}}}{(\breve{\gamma}-1)^{\frac{C_{h+1}}{\sum_{l=1}^{h+1}C_1}-1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\frac{(\breve{\gamma}^{\Phi^{\breve{q}}\varrho_{h+1}}-1)^{\frac{C_{h+1}}{\sum_{l=1}^{h+1}C_1}}}{(\breve{\gamma}-1)^{\frac{C_{h+1}}{\sum_{l=1}^{h+1}C_1}-1}}\right)\right)^{\frac{1}{\breve{q}}}\right\}$$

$$=\left(1-\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{h+1}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{h+1}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{h+1}\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{h+1}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{h+1}\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{h+1}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\left(\therefore\sum_{l=1}^{h+1}\frac{C_1}{\sum_{l=1}^{h+1}C_1}\right)$$

Thus when we assume the assertion is correct for $p = h$ then it will be accurate for $h+1$. Hence, through the procedure of mathematical induction, the provided statement holds true and is accurate for all positive integers p.

Theorem 2 (Idempotent axiom) If all the $\breve{q}$-rung picture fuzzy numbers $\varrho_l = (\Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l})(l = 1, 2, \ldots, p)$ are alike, i.e., $\varrho_1 = \varrho\forall l$, $l = 1, 2, \ldots, p$, then $\breve{q}$-RPFPWFA$(\varrho_1, \varrho_2, \ldots, \varrho_p) = \varrho$.

Proof. since $\varrho_1 = \varrho, \forall l$, then we get
$\breve{q}$ − RPFPWFA$(\varrho_1, \varrho_2, \ldots, \varrho_p)$

$$=\left\{\left(1-\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right\}$$

$$=\left\{\left(1-\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho}^{\breve{q}}}-1\right)^{\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)$$

$$=\left\{\left(1-\log_{\breve{\gamma}}\left(1+\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\sum_{l=1}^{p}\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1+\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)^{\sum_{l=1}^{p}\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\sum_{l=1}^{p}\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right\}$$

$$=\left\{\left(1-\log_{\breve{\gamma}}\left(1+\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1+\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)\right)\right)^{\frac{1}{\breve{q}}}, \left(\log_{\breve{\gamma}}\left(1+\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\sum_{l=1}^{p}\frac{C_1}{\sum_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right\}$$

$$=\left((\Psi_{\varrho}^{\breve{q}})^{\frac{1}{\breve{q}}}, (\Theta_{\varrho}^{\breve{q}})^{\frac{1}{\breve{q}}}, (\Phi_{\varrho}^{\breve{q}})^{\frac{1}{\breve{q}}}\right)$$

$$=\{\Psi_{\varrho}, \Theta_{\varrho}, \Phi_{\varrho}\}$$

$$=\varrho$$

Theorem 3 (Boundedness axiom) Let $\varrho_l = (\Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l})$ $(l = 1, 2, \ldots, p)$ be a set $\breve{q}$-rung picture fuzzy numbers. Suppose $\varrho^- = min(\varrho_1, \varrho_2, .., \varrho_p)$ and $\varrho^+ = max(\varrho_1, \varrho_2, .., \varrho_p)$ then $\varrho^- \leq \breve{q}$-RPFPWFWA$(\varrho_1, \varrho_2, .., \varrho_p) \leq \varrho^+$.

Proof. Let $\varrho^- = (\Psi^-, \Theta^-, \Phi^-)$ and $\varrho^+ = (\Psi^+, \Theta^+, \Phi^+)$. Thus $\Psi^- = min_1\Psi_{\varrho_1}, \Theta^- = max_1\Theta_{\varrho_1}, \Phi^- = max_1\Phi_{\varrho_1}$ and $\Psi^+ = max_1\Psi_{\varrho_1}, \Theta^+ = min_1\Theta_{\varrho_1}, \Phi^+ = min_1\Phi_{\varrho_1}$.

$$\therefore\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-(\Psi^{-})^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\leq$$
$$\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\leq$$
$$\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-(\Psi^{+})^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$

In a similar manner,

$$\therefore\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{(\Theta^{+})^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\leq$$
$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\leq$$
$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{(\Theta^{-})^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$

and

$$\therefore\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{(\Phi^{+})^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\leq$$
$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\leq$$
$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-(\Phi^{-})^{\breve{q}}}\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$

Hence, $\varrho^{-}\leq\breve{q}$-RPFPWFA$(\varrho_1,\varrho_2,\ldots,\varrho_p)\leq\varrho^{+}$.

Theorem 4 (Monotone axiom) Let $\varrho_1'=(\Psi_{\varrho_1'},\Theta_{\varrho_1'},\Phi_{\varrho_1'})$ $(l=1,2,\ldots,p)$ and Let $\varrho_1=(\Psi_{\varrho_1},\Theta_{\varrho_1},\Phi_{\varrho_1})$ $(l=1,2,\ldots,p)$ be two families of $\breve{q}$-rung picture fuzzy numbers. Suppose $\Psi_{\varrho_1}\leq\Psi_{\varrho_1'}$, $\Theta_{\varrho_1}\geq\Theta_{\varrho_1'}$ and $\Phi_{\varrho_1}\leq\Phi_{\varrho_1'}$, $\forall l$, then

$\breve{q}$-RPFPWFA$(\varrho_1,\varrho_2,\ldots,\varrho_p)\leq\breve{q}$-RPFPWFA$(\varrho_1',\varrho_2',\ldots,\varrho_p')$.

Proof. If $\Psi_{\varrho_1}\leq\Psi_{\varrho_1'}$, $\Theta_{\varrho_1}\geq\Theta_{\varrho_1'}$ and $\Phi_{\varrho_1}\leq\Phi_{\varrho_1'}$, $\forall(l=1,2,\ldots,p)$, then

$$\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}}C_1}\geq\left(\breve{\gamma}^{1-\Psi_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}}C_1}$$
$$\Rightarrow\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\geq$$
$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$
$$\Rightarrow\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\leq$$
$$\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$

Similarly, we can show that

$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\geq$$
$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$

and

$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\geq$$
$$\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}$$

Therefore,

$$\left\{\left(\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Pi_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^{\breve{q}}-\right.$$
$$\left(\left(\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Pi_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^{\breve{q}}-\right.$$
$$\left.\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Pi_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^{\breve{q}}\right\}\leq$$
$$\left\{\left(\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Pi_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^{\breve{q}}-\right.$$
$$\left(\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^{\breve{q}}-$$
$$\left.\left(\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho_1'}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^{\breve{q}}\right\}$$

In particular,

$$\left\{2+\left(\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{2}}\right)^{2}-\right.$$
$$\left(\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{2}}\right)^{2}-$$
$$\left.\left(\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi_{\varrho_1}^{\breve{q}}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^{2}\right\}\Big/3\leq$$

$$\left\{2+\left(\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi^{\breve{q}}_{\varrho'_1}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{2}}\right)^2-\left(\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta^{\breve{q}}_{\varrho'_1}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{2}}\right)^2-\left(\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi^{\breve{q}}_{\varrho'_1}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right)^2\right\}\Big/3$$

Let us denote $\varrho = \breve{q}$-RPFPWFA$(\varrho_1, \varrho_2, ..., \varrho_p)$ and $\varrho = \breve{q}$-RPFPWFA$(\varrho'_1, \varrho'_2, ..., \varrho'_p)$. Then by definition 3, we will have $S_\varrho \le S_{\varrho'}$

We obtain, $\varrho = \breve{q}$-RPFPWFA$(\varrho_1, \varrho_2, ..., \varrho_p) \le \varrho = \breve{q}$-RPFPWFA$(\varrho'_1, \varrho'_2, ..., \varrho'_p)$

3.1.2 $\breve{q}$-Rung Picture Fuzzy Prioritized Weighted Frank Geometric ($\breve{q}$-RPFPWFG) Aggregator

In this segment, $\breve{q}$-rung picture fuzzy prioritized weighted Frank geometric ($\breve{q}$-RPFPWFG) aggregator is proposed and its characteristics are proved.

Definition 8 For a set of $\breve{q}$-RPFNs denoted as ϱ_1, with each ϱ_1 being represented as, $\varrho_l = \langle\Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l}\rangle$ for $l = (1, 2, ..., p)$ the $\breve{q}$-RPFPWFG aggregator is formally defined through a mapping denoted as $\breve{q}$-RPFPWFG: $\varrho^p \to \varrho$, where

$$\breve{q}\text{-RPFPWFG}(\varrho_1, \varrho_2, ..., \varrho_p) = \bigotimes_{l=1}^{p}\left(\varrho_l^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)$$

$$= \left(\varrho_1^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\bigotimes \varrho_2^{\frac{C_2}{\Sigma_{l=1}^{p}C_1}}\bigotimes ... \bigotimes \varrho_p^{\frac{C_p}{\Sigma_{l=1}^{p}C_1}}\right)$$

where, $C_1 = \prod_{m=1}^{l-1} S(\varrho_m), l = (1, 2, .., p)$ with $C_1 = 1$ and S_{ϱ_m} signifies the score associated with ϱ_m.

Theorem 5 The $\breve{q}$-RPFPWFG aggregator is used to combine a collection of $\breve{q}$-RPFNs represented by $\varrho_l = \langle\Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l}\rangle$ for $l = (1, 2, ..., p)$. The combined value of these $\breve{q}$-RPFNs by applying the $\breve{q}$-RPFPWFG aggregator is another $\breve{q}$-RPFN. The $\breve{q}$-RPFPWFG aggregator is provided below:

$$\breve{q}\text{-RPFPWFG}(\varrho_1, \varrho_2, ..., \varrho_p) = \bigotimes_{l=1}^{p}\left(\varrho_l^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)$$

$$=\left\{\left(log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{\Psi^{\breve{q}}_{\varrho_1}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Theta^{\breve{q}}_{\varrho_1}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)\right)^{\frac{1}{\breve{q}}}, \left(1-log_{\breve{\gamma}}\left(1+\prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Phi^{\breve{q}}_{\varrho_1}}-1\right)^{\frac{C_1}{\Sigma_{l=1}^{p}C_1}}\right)\right)^{\frac{1}{\breve{q}}}\right\} \quad (4)$$

where, $C_1 = \prod_{m=1}^{l-1} S(\varrho_m), l = (2, ..., p)$ with $C_1 = 1$ and $S(\varrho_m)$ signifies the score associated with ϱ_m.

Proof. The proof of this theorem is similar to that of Theorem 1.

Theorem 6 (Idempotent axiom) If all the $\breve{q}$-rung picture fuzzy numbers $\varrho_l = \langle\Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l}\rangle$ $l = (1, 2, ..., p)$ are alike, i.e., $\varrho_l = \varrho \forall l, l = 1, 2, ..., p$, then $\breve{q}$-RPFPWFG$(\varrho_1, \varrho_2, ..., \varrho_p)$ $= \varrho$.

Proof. The proof of this theorem is similar to that of Theorem 2.

Theorem 7 (Boundedness axiom) Let $\varrho_l = \langle\Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l}\rangle$ $l = (1, 2, ..., p)$ be a set of $\breve{q}$-rung picture fuzzy numbers. Suppose $\varrho^- = min(\varrho_1, \varrho_2, ..., \varrho_p)$ and $\varrho^+ = max(\varrho_1, \varrho_2, ..., \varrho_p)$ then $\varrho^- \le \breve{q}$-RPFPWFG$(\varrho_1, \varrho_2, ..., \varrho_p) \le \varrho^+$.

Proof. The theorem's proof is similar to that of Theorem 3.

Theorem 8 (Monotone axiom) Let $\varrho'_l = \left(\Psi_{\varrho'_l}, \Theta_{\varrho'_l}, \Phi_{\varrho'_l}\right)$ $l = (1, 2, ..., p)$ and $\varrho_l = \left(\Psi_{\varrho_l}, \Theta_{\varrho_l}, \Phi_{\varrho_l}\right)$ $l = (1, 2, ..., p)$ be two families of -rung picture fuzzy numbers. Suppose $\Psi_{\varrho_l} \le \Psi_{\varrho'_l}, \Theta_{\varrho_l} \ge \Theta_{\varrho'_l}$ and $\Phi_{\varrho_l} \ge \Phi_{\varrho'_l}, \forall l$, then

$\breve{q}$-RPFPWFG$(\varrho_1, \varrho_2, ..., \varrho_p) \le \breve{q}$-RPFPWFG$(\varrho'_1, \varrho'_2, ..., \varrho'_p)$

Proof. The theorem's proof is similar to that of Theorem 4.

3.2 Unique MAGDM Procedure Based on Devised Aggregators

This study applies the $\breve{q}$-RPFPWFA and $\breve{q}$-RPFPWFG operators in the context of MAGDM technique. The study considers a set of alternatives (choices) $G_1, G_2, ..., G_s$ and a collection of attributes $R_1, R_2, ..., R_k$. The prioritization amidst the attributes can be indicated by the ordering, $R_1 \succ R_2 \succ ... \succ R_k$ where the attribute is R_l more important than R_m if $l < m$. Similarly, if we have a group of experts or decision-makers say $M_1, M_2, ..., M_v$ then the prioritization amidst the experts can be indicated by the ordering, $M_1 \succ M_2 \succ ... \succ M_v$ where M_p is more important than M_q if $p < q$. Moreover the $\breve{q}$-rung picture fuzzy decision matrix ($\breve{q}$-RPFDM) $M^b = (\epsilon^b_{rc})_{s\times k} = (\Psi^b_{rc}, \Theta^b_{rc}, \Phi^b_{rc})_{s\times k}$ and the $\breve{q}$-RPFN, denoted as $(\epsilon^b_{rc})_{s\times k} = (\Psi^b_{rc}, \Theta^b_{rc}, \Phi^b_{rc})$, is assigned by experts. In this context, $\Psi^b_{rc}, \Theta^b_{rc}$, and Φ^b_{rc} indicates the PG, IG and NG of the choices with regard to the attribute such that the condition, $0 \le (\Psi_{rc})^{\breve{q}} + (\Theta_{rc})^{\breve{q}} + (\Phi_{rc})^{\breve{q}} \le 1$ holds true for all cases where r ranges from $(r = 1, 2, ..., s)$ and c ranges from $(c = 1, 2, ..., k)$. Essentially, there are two categories of criteria: beneficial type(B) where a higher value is preferred and cost type (C) where a lower value is preferred. To make a fair comparison, we need to standardize these criteria into uniform type. Thus, we need to transform the original $\breve{q}$-RPFDM $M^b = (\epsilon^b_{rc})_{s\times k}$ into a normalized $\breve{q}$-RPFDM $\overline{M}^b = (\epsilon^b_{rc})_{s\times k}$, where

Table 1. $\breve{q}$-RPFDM $M^1 = (\epsilon^1_{rc})$ Acquired from Expert 1

	R_1	R_2	R_3	R_4
G_1	⟨0.7, 0.3, 0.5⟩	⟨0.9, 0.2, 0.3⟩	⟨0.5, 0.4, 0.3⟩	⟨0.1, 0.6, 0.4⟩
G_2	⟨0.4, 0.6, 0.1⟩	⟨0.6, 0.4, 0.3⟩	⟨0.6, 0.6, 0.3⟩	⟨0.8, 0.5, 0.3⟩
G_3	⟨0.7, 0.5, 0.4⟩	⟨0.5, 0.4, 0.2⟩	⟨0.8, 0.2, 0.2⟩	⟨0.7, 0.5, 0.2⟩
G_4	⟨0.8, 0.2, 0.1⟩	⟨0.6, 0.5, 0.5⟩	⟨0.9, 0.2, 0.1⟩	⟨0.9, 0.1, 0.1⟩

Table 2. $\breve{q}$-RPFDM $M^2 = (\epsilon^2_{rc})$ Acquired from Expert 2

	R_1	R_2	R_3	R_4
G_1	⟨0.5, 0.3, 0.3⟩	⟨0.8, 0.1, 0.2⟩	⟨0.6, 0.5, 0.3⟩	⟨0.3, 0.5, 0.4⟩
G_2	⟨0.3, 0.6, 0.1⟩	⟨0.6, 0.4, 0.2⟩	⟨0.6, 0.4, 0.3⟩	⟨0.8, 0.4, 0.3⟩
G_3	⟨0.6, 0.5, 0.4⟩	⟨0.5, 0.3, 0.2⟩	⟨0.9, 0.2, 0.1⟩	⟨0.7, 0.5, 0.1⟩
G_4	⟨0.8, 0.2, 0.1⟩	⟨0.7, 0.5, 0.4⟩	⟨0.8, 0.3, 0.2⟩	⟨0.8, 0.2, 0.1⟩

$$\bar{M}^b = \begin{cases} \epsilon^b_{rc} & \text{if } c \in B \\ (\epsilon^b_{rc})^c & \text{if } c \in C \end{cases} \tag{5}$$

where (ϵ^b_{rc}) complement is denoted as $(\epsilon^b_{rc})^c$ and is defined $\ni (\epsilon^b_{rc})^c = \langle \Phi^b_{rc}, \Theta^b_{rc}, \Psi^b_{rc} \rangle$. In this research, MULTIMOORA technique utilized for $\breve{q}$-rung orthopair fuzzy sets in Aydemir and Yilmaz Gündüz (2020) will be enhanced to $\breve{q}$-RPFS.

3.2.1 Mathematical Formulation of Enhanced MULTIMOORA Based Upon $\breve{q}$-Rung Picture Fuzzy Prioritized Weighted Aggregators

The MULTIMOORA method comprises various Moora methods and a multiplicative utility function. These methods encompass an additive utility function known as the ratio system (RS), a reference point (RP) approach, and a multiplicative utility function referred to as the full multiplicative form (MF), as introduced by Brauers and Zavadskas (2010). We employ the proposed aggregators. This approach allows us to effectively capture the attributes due to the flexibility of the Frank operators and the prioritized aggregator's parameters. The $\breve{q}$-RPF MULTIMOORA MAGDM method with unspecified weight information can be elucidated through the following steps and its stages:

Step 1: The decision values provided by experts would be in the context of $\breve{q}$-RPFDM and suppose the attributes are of different nature, then the normalization of $\breve{q}$-RPFDM will be done using Equation (5).

Step 2: As the weight information is unspecified for experts, determine the prioritized weights, with the help of, $B^b_{rc} = \prod_{l=1}^{b-1} S(M^l_{rc}), b = (2, \ldots, v)$, with the condition that $B^1_{rc} = 1$.

Step 3: To combine all the $\breve{q}$-RPFDMs in to single $\breve{q}$-RPFDM, utilise $\breve{q}$-RPFPWFA or $\breve{q}$-RPFPWFG aggregators given in Equations (3) and (4).

$M_{rc} = \breve{q}\text{-RPFPWFA}(M^1_{rc}, M^2_{rc}, \ldots M^v_{rc})$

$$= \Bigg\{ \left(1 - log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B^b_{rc}}{\sum_{b=1}^{v} B^b_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\left(log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B^b_{rc}}{\sum_{b=1}^{v} B^b_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B^b_{rc}}{\sum_{b=1}^{v} B^b_{rc}}}\right)\right)^{\frac{1}{\breve{q}}} \Bigg\} \tag{6}$$

or

$M_{rc} = \breve{q}\text{-RPFPWFG}(M^1_{rc}, M^2_{rc}, \ldots M^v_{rc})$

$$= \Bigg\{ \left(log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{\Psi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B^b_{rc}}{\sum_{b=1}^{v} B^b_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(1 - log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Theta^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B^b_{rc}}{\sum_{b=1}^{v} B^b_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(1 - log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Phi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B^b_{rc}}{\sum_{b=1}^{v} B^b_{rc}}}\right)\right)^{\frac{1}{\breve{q}}} \Bigg\} \tag{7}$$

Step 4: Compute the prioritized matrix by, $B_{rc} = \prod_{j=1}^{c-1} S(M_{rj}), c = (2, \ldots, k)$, with the condition that $B_{r1} = 1$.

Step 5: This step consists of three stages.

Stage (i) : In this stage, based on the values of the prioritized matrix we do apply $\breve{q}$-RPFPWFA aggregator to the ratio system within the MULTIMOORA $\breve{q}$-RPFS framework. We then determine the $\breve{q}$-RPF utility for each alternative using the

Table 3. $\breve{q}$-RPFDM $M^3 = (\epsilon^3_{rc})$ Acquired from Expert 3

	R_1	R_2	R_3	R_4
G_1	$\langle 0.6, 0.3, 0.2\rangle$	$\langle 0.8, 0.2, 0.3\rangle$	$\langle 0.5, 0.5, 0.2\rangle$	$\langle 0.2, 0.5, 0.3\rangle$
G_2	$\langle 0.4, 0.5, 0.2\rangle$	$\langle 0.7, 0.3, 0.2\rangle$	$\langle 0.6, 0.4, 0.4\rangle$	$\langle 0.8, 0.4, 0.4\rangle$
G_3	$\langle 0.8, 0.3, 0.3\rangle$	$\langle 0.5, 0.4, 0.4\rangle$	$\langle 0.8, 0.2, 0.3\rangle$	$\langle 0.7, 0.5, 0.3\rangle$
G_4	$\langle 0.8, 0.3, 0.2\rangle$	$\langle 0.7, 0.5, 0.5\rangle$	$\langle 0.9, 0.3, 0.1\rangle$	$\langle 0.8, 0.2, 0.2\rangle$

Table 4. Combined $\breve{q}$-RPFDM $M = (\epsilon_{rc})$ for $\breve{q} = 3$ and $\breve{\gamma} = 2$

	R_1	R_2	R_3	R_4
G_1	$\langle 0.63, 0.3, 0.35\rangle$	$\langle 0.85, 0.16, 0.26\rangle$	$\langle 0.54, 0.45, 0.28\rangle$	$\langle 0.21, 0.55, 0.38\rangle$
G_2	$\langle 0.37, 0.58, 0.11\rangle$	$\langle 0.63, 0.37, 0.24\rangle$	$\langle 0.6, 0.49, 0.32\rangle$	$\langle 0.8, 0.44, 0.32\rangle$
G_3	$\langle 0.7, 0.42, 0.4\rangle$	$\langle 0.5, 0.4, 0.4\rangle$	$\langle 0.5, 0.37, 0.23\rangle$	$\langle 0.84, 0.2, 0.18\rangle$
G_4	$\langle 0.8, 0.22, 0.12\rangle$	$\langle 0.66, 0.5, 0.47\rangle$	$\langle 0.87, 0.26, 0.13\rangle$	$\langle 0.85, 0.16, 0.12\rangle$

following formula:

$$=\left\{\left(1 - log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Psi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B_{rc}}{\sum_{c=1}^{k} B_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\left(log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{\Theta^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B_{rc}}{\sum_{c=1}^{k} B_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{\Phi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B_{rc}}{\sum_{c=1}^{k} B_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}\right\} \quad (8)$$

$= \langle \Psi^{RS}_r, \Theta^{RS}_r, \Phi^{RS}_r\rangle, (r = 1, 2, ..., s)$

The score function is utilized on the results obtained in the preceding step.

$$\breve{\xi}^{RS}_r = (2+(\Psi^{RS}_r)^2-(\Theta^{RS}_r)^2-(\Phi^{RS}_r)^2)/3, (r = 1, 2, ..., s) \quad (9)$$

The process of normalizing crisp values is performed as follows:

$$\bar{\xi}^{RS}_r = \frac{\breve{\xi}^{RS}_r}{max_r\breve{\xi}^{MF}_r}, (r = 1, 2, ..., s) \quad (10)$$

Ultimately, we attain the highest utility score for $\bar{\xi}^{RS}_r$.

Stage (ii): During this stage, we employ the reference point approach. We determine the reference point (e^*_c) and subsequently compute the Chebyshev distance for each available choices. The reference point will be found from the input values of Equation (8) is elucidated as follows,

$e^*_c = (\Psi^*_c, \Theta^*_c, \Phi^*_c,); \Psi^*_c = max\Psi_{rc}, \Theta^*_c = min\Theta_{rc}, \Phi^*_c = min\Phi_{rc}, c = (1, ..., k)$

Then we employed the Hamming distance as the distance metric. Let us consider $\varrho_1 = (\Psi_1, \Theta_1, \Phi_1, \Upsilon_1)$ and $\varrho_2 = (\Psi_2, \Theta_2,$
$\Phi_2, \Upsilon_2)$ as the two $\breve{q}$-RPFNs, (where Υ indicates the refusal grade) then,

$$d_{hamming}(\varrho_1, \varrho_2) = \left(|\Psi^q_1 - \Psi^q_2| + |\Theta^q_1 - \Theta^q_2| + |\Phi^q_1 - \Phi^q_2| + |\Upsilon^q_1 - \Upsilon^q_2|\right)/2 \quad (11)$$

$$d_{rc} = d_{hamming}(\epsilon_{rc}, e^*_c), (r = 1, 2, ..., s), (c = 1, 2, .., k) \quad (12)$$

Moreover, we compute the maximum Chebyshev distance from the reference point for each choice.

$$\xi^{RP}_r = max_c d_{rc}$$

It's important to note that the reference point follows a non-compensatory approach, meaning that a lower ξ^{RP}_r value indicates higher utility. However, in the subsequent step, we can determine normalized utility scores as follows:

$$\bar{\xi}^{RP}_r = \frac{min_r\xi^{RP}_r}{\xi^{RP}_r}, (r = 1, 2, ..., s) \quad (13)$$

Ultimately, we attain the highest utility score for $\bar{\xi}^{RP}_r$.

Stage (iii): At this stage, the $\breve{q}$-RPFPWFG operator, as defined in Equation (14), is applied to the multiplicative utility function (MF). The process involves computing the $\breve{q}$-RPF utility for each choice, as follows:

$$\xi^{MF}_r = \breve{q}\text{-RPFPWFG}(M_{r1}, M_{r2}, ..., M_{rk})$$

$$=\left\{\left(log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{\Psi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B_{rc}}{\sum_{c=1}^{k} B_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(\left(1 - log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Theta^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B_{rc}}{\sum_{c=1}^{k} B_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}, \left(1 - log_{\breve{\gamma}}\left(1 + \prod_{l=1}^{p}\left(\breve{\gamma}^{1-\Phi^{\breve{q}}_{\varrho_1}} - 1\right)^{\frac{B_{rc}}{\sum_{c=1}^{k} B_{rc}}}\right)\right)^{\frac{1}{\breve{q}}}\right\} \quad (14)$$

$= \langle \Psi^{MF}_r, \Theta^{MF}_r, \Phi^{MF}_r\rangle, (r = 1, 2, ..., s)$

The score function is utilized on the results obtained in the preceding step.

$$\breve{\xi}^{MF}_r = (2+(\Psi^{MF}_r)^2-(\Theta^{MF}_r)^2-(\Phi^{MF}_r)^2)/3, (r = 1, 2, ..., s)$$

Table 5. Outcome of MULTIMOORA as per the Dominance Theory

Candidates	Ratio System (RS)	Reference Point (RP)	Multiplicative Form (MF)	MULTIMOORA (MM)
G_1	3	4	3	3
G_2	4	2	4	4
G_3	2	3	2	2
G_4	1	1	1	1

Table 6. Impact of Various Parameters on MULTIMOORA $\breve{q}$-RPFNs

$\breve{q}$	$\breve{\gamma}$	Ratio System (RS)	Multiplicative Form (MF)	MULTIMOORA (MM)
$\breve{q}=3$	$\breve{\gamma}=3$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=5$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=8$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=12$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=15$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
$\breve{q}=4$	$\breve{\gamma}=3$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=5$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=8$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=12$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=15$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
$\breve{q}=5$	$\breve{\gamma}=3$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=5$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=8$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=12$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$
	$\breve{\gamma}=15$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$	$G_4 > G_3 > G_1 > G_2$

(15)

The process of normalizing crisp values is performed as follows:

$$\overline{\xi}_r^{MF} = \frac{\breve{\xi}_r^{MF}}{max_r \breve{\xi}_r^{MF}}, (r = 1, 2, \ldots, s) \tag{16}$$

Step 6: Ultimately, we attain the highest utility score for ξ_r^{MF}. Dominance theory elucidated by Brauers et al. (2012) is utilized to assess the rankings derived from three Moora methods specified in Equations (10), (13), and (16).

3.3 Numerical Instance

In this section, we present a numerical instance using the MULTIMOORA(MM) $\breve{q}$-RPF method to showcase the reliability and uniformity of the suggested approaches. Also we examine the influence of parameters $\breve{q}$ and $\breve{\gamma}$ (Frank parameter) for the illustration. We then compare the outcomes obtained by our aggregators with those available operators in the field.

3.3.1 The Decision-Making Procedure within the $\breve{q}$-rung Picture Fuzzy Framework

Akram et al. (2020) A housing society is in the process of selecting a proficient and reliable Financial Director. To facilitate this decision, the society has designated three key experts: M_1 - A chartered accountant, M_2 - Residential Community Owner and M_3 - Treasurer of the housing society. These decision makers have been prioritized in the following order:

$M_1 > M_2 > M_3$, which means that M_1 holds a higher level of priority compared to the others. It's important to note that the appointment process is devoid of any political or external influence. The interview panel has rigorously assessed four candidates G_1, G_2, G_3 and G_4 for the Financial Director position, following a preliminary screening. The evaluation was based on the following four attributes:

1. Effective Communication;
2. Previous Exposure;
3. Academic Record and
4. Proficiency.

The attribute R_1 holds a higher priority compared to the other attributes, resulting in the prioritization sequence of $R_1 > R_2 > R_3 > R_4$.

Step 1: The experts have provided decision values in the $\breve{q}$-RPFNs context, and since all the attributes are considered to be of a beneficial nature, there is no requirement for normalization. Thus the expert's decision values are displayed as $\breve{q}$-RPFDM in Tables 1 - 3.

Step 2: We need to evaluate the prioritized weights since the weight information is unspecified for experts. So B_{rc}^b values ($b = 1, 2, 3$) can be determined as specified in the procedure,

Table 7. Analysis in Comparison to Other Methods

Aggregators	$S(G_1)$	$S(G_2)$	$S(G_3)$	$S(G_4)$	Definitive Position	Ideal Choice
$SFWA$ (2019)	0.6794	0.5941	0.6439	0.7368	$G_4 > G_3 > G_1 > G_2$	G_4
$SFWG$ (2019)	0.6313	0.5763	0.6267	0.7086	$G_4 > G_3 > G_1 > G_2$	G_4
$SFPWA$ (2020)	0.7471	0.6873	0.7462	0.8384	$G_4 > G_3 > G_1 > G_2$	G_4
$SFPWG$(2020)	0.6807	0.6227	0.7081	0.8106	$G_4 > G_3 > G_1 > G_2$	G_4
Proposed Aggregators						
(i) RS with $\check{q}$-RPFPWFA	0.7564	0.6911	0.7652	0.8423	$G_4 > G_3 > G_1 > G_2$	G_4
(ii) MF with $\check{q}$-RPFPWFG	0.6871	0.6496	0.7291	0.7918	$G_4 > G_3 > G_1 > G_2$	G_4
Cumulative result of MULTIMOORA by Dominance theory					$G_4 > G_3 > G_1 > G_2$	G_4

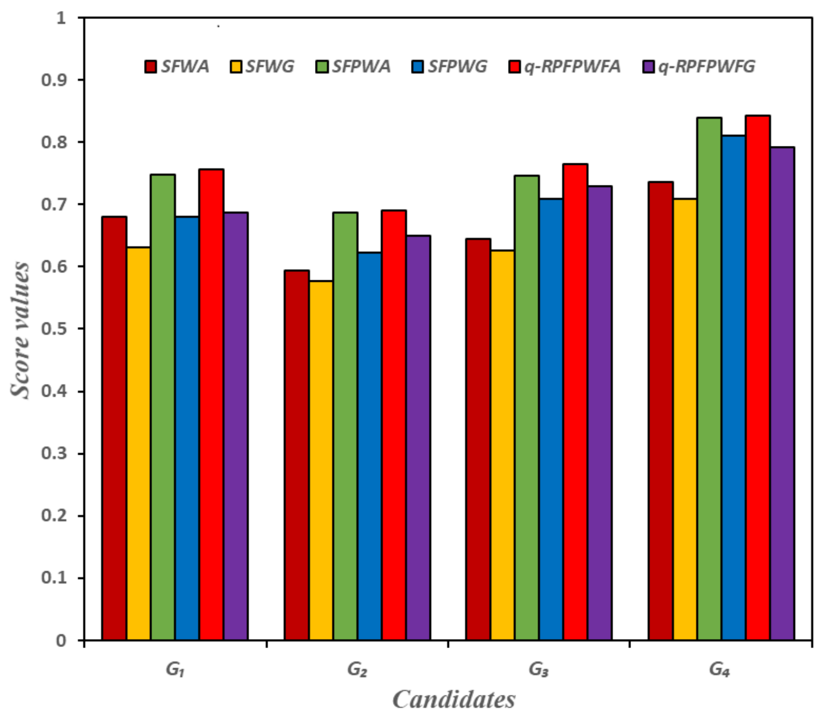

Figure 1. Contrasting the Proposed Approach with Few Existing Techniques

$$B^1_{sk} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

Now to determine the values of B^b_{rc} where b varies from 1 to v, utilize the equation below: $B^b_{rc} = \prod_{l=1}^{b-1} S(M^1_{rc})$, $(b = 2, \ldots, v)$ with the condition that $B^b_{rc} = 1$. Thus,

$$B^2_{sk} = \begin{pmatrix} 0.7167 & 0.8933 & 0.6667 & 0.4967 \\ 0.5967 & 0.7033 & 0.6367 & 0.7667 \\ 0.6933 & 0.6833 & 0.8533 & 0.7333 \\ 0.8633 & 0.6200 & 0.9200 & 0.9300 \end{pmatrix}$$

and

$$B^3_{sk} = \begin{pmatrix} 0.4945 & 0.7712 & 0.4489 & 0.2781 \\ 0.3421 & 0.5064 & 0.4478 & 0.6108 \\ 0.4714 & 0.4829 & 0.7850 & 0.5451 \\ 0.7453 & 0.4298 & 0.7698 & 0.8029 \end{pmatrix}$$

Step 3: Combine the three $\check{q}$-RPF decision matrices ($M^b = (\epsilon^b_{rc})_{s\times k}$) given in Tables 1-3 into single matrix ($M = (\epsilon_{rc})_{s\times k}$) by utilising Equation (6) or (7). Here, we employ $\check{q}$-RPFPWFA aggregator given in the Equation (6).

Step 4: Compute the prioritized matrix $B_{rc}(r = 1, 2, ..., s;$ $c = 1, 2, ..., k)$ by utilizing the equation $B_{rc} = \prod_{j=1}^{c-1} S(M_{rj})$ for $c = 2, ..., k$, with the condition that $B_{r1} = 1$, as described in the methodology.

$$B_{sk} = \begin{pmatrix} 1 & 0.7281 & 0.6381 & 0.4277 \\ 1 & 0.5951 & 0.4376 & 0.2943 \\ 1 & 0.7179 & 0.4930 & 0.4327 \\ 1 & 0.8591 & 0.5626 & 0.5012 \end{pmatrix}$$

Step 5: This process comprises three distinct stages.

Stage (i) In this stage, the $\check{q}$-RPFPWFA aggregator given in Equation (8) is applied to the ratio system in the context of MULTIMOORA-$\check{q}$-RPFS to merge the ϵ_{rc} values in Table 4. The $\check{q}$-RPF utility for each alternative is determined as outlined below.

$\xi^{RS}_1 = \langle 0.6782, 0.3073, 0.3104\rangle; \xi^{RS}_2 = \langle 0.5880, 0.4859, 0.1905\rangle; \xi^{RS}_3 = \langle 0.7034, 0.3620, 0.2609\rangle$; and $\xi^{RS}_4 = \langle 0.7975, 0.2747, 0.1831\rangle$;

The score function specified in Equation (9) is utilized on the results obtained in the preceding step.

$\bar{\xi}^{RS}_r = \langle 0.7564; 0.6911; 0.7652; 0.8423\rangle, (r = 1, 2, 3, 4)$

Then the process of normalizing crisp values is performed as follows:

$$\bar{\xi}^{-RS}_r = \frac{\bar{\xi}^{-RS}_r}{\max_r \bar{\xi}^{-RS}_r}, (r = 1, 2, 3, 4).$$

$$\bar{\xi}^{-RS}_r = (0.8980, 0.8205, 0.9085, 1).$$

The outcome generated by the MULTIMOORA $\check{q}$-RPF ratio system is ranked as:

$G_4 \succ G_3 \succ G_1 \succ G_2$

Stage (ii): During this stage, the reference point is established, and then the Chebyshev distance is computed using the Equations (11) and (12) for every available choice.

$e^*_c = (\Psi^*_c, \Theta^*_c, \Phi^*_c); \Psi^*_c = max_r\Psi_{rc}, \Theta^*_c = min_r\Phi_{rc}, \Psi^*_c = min_r\Phi_{rc}, (c = 1, ..., k)$

$d_{rc} = d_{hamming}(\epsilon_{rc}, e^*_c), (r = 1, 2, 3, 4), (c = 1, 2, 3, 4).$

Additionally, the highest Chebyshev distance originating from the reference point is computed for each available choice.

$\xi^{RP}_r = max_r d_{rc} = (0.6049, 0.4617, 0.4945, 0.3266)$

The reference point follows a non-compensatory approach, where a lower ξ^{RP} value implies higher utility. Subsequently, normalized utility scores can be determined by,

$$\bar{\xi}^{-RP}_r = \frac{min_r \xi^{RP}_r}{\xi^{RP}_r} = (0.5400, 0.7074, 0.6605, 1), (r = 1, 2, 3, 4).$$

The outcome generated by the MULTIMOORA $\check{q}$-RPF reference point is ranked as:

$G_4 \succ G_2 \succ G_3 \succ G_1$

Stage (iii): At this stage, the $\check{q}$-RPFPWFG operator, as defined in Equation (14), is applied to the multiplicative utility function in the context of MULTIMOORA–RPFS to merge the values in Table 4. The process involves computing the $\check{q}$-RPF utility for each choice as follows:

$\xi^{MF}_1 = \langle 0.5617, 0.3892, 0.3206\rangle; \xi^{MF}_2 = \langle 0.5175, 0.5090, 0.2445\rangle; \xi^{MF}_3 = \langle 0.6650, 0.3982, 0.3105\rangle$; and $\xi^{MF}_4 = \langle 0.7773, 0.3572, 0.3180\rangle$;

The score function specified in Equation (15) is utilized on the results obtained in the preceding step.

$\bar{\xi}^{MF}_r = \langle 0.6871; 0.6496; 0.7291; 0.7918\rangle, (r = 1, 2, 3, 4)$

Then the process of normalizing crisp values is performed as follows:

$$\bar{\xi}^{-MF}_r = \frac{\bar{\xi}^{-MF}_r}{\max_r \bar{\xi}^{-MF}_r}, (r = 1, 2, 3, 4).$$

$$\bar{\xi}^{-MF}_r = (0.8678, 0.8205, 0.9208, 1).$$

The outcome generated by the MULTIMOORA $\check{q}$-RPF ratio system is ranked as:

$G_4 \succ G_3 \succ G_1 \succ G_2$

Dominance theory is employed to evaluate the rankings derived from both the Moora and Multiplicative Form methods. Specifically, the rankings for MULTIMOORA with $\check{q} = 3$ and $\check{\gamma} = 2$ can be found in Table 5, with the fourth candidate (G_4) emerging as the most favourable choice. The ranking from the MULTIMOORA framework is as follows:

$G_4 \succ G_3 \succ G_1 \succ G_2$

3.4 Impact of the Parameters and on MULTIMOORA $\check{q}$-RPFS

Studies on $\check{q}$-RPF have delved into MADM problems from a wider angle, taking $\check{q} = 3$ in the example above. It is significant to highlight that with higher $\check{q}$ values, the score values tend to approach convergence. Now, let us delve deeper into the influences of the variables, $\breve{\gamma}$ and $\breve{q}$ on the proposed methods in more detail. In the reference point method, where no aggregation method is employed, it solely varies with respect to the $\breve{q}$ values. Consequently, Table 6 examines various combinations of $\breve{q}$ and $\breve{\gamma}$ parameters using the MULTIMOORA procedure. The $\breve{q}$-RPFPWFA ratio system and $\breve{q}$-RPFPWFG multiplicative form is applied in this analysis. These two techniques of MULTIMOORA are combined into a unified outcome based on the dominance theory for $\breve{q}$-RPFNs. Thus, based on the aforementioned MULTIMOORA techniques, the ordering of these four candidate choices is obtained such that, G_4 is preferred over G_3, which, in turn, is favoured over G_1, and finally, G_2 is the least preferred. Consequently, regardless of different values of the variable $\breve{\gamma}(\breve{\gamma} = 3, 5, 8, 12$ and $15)$ and for a fixed $\breve{q}(\breve{q} = 3)$, G_4 emerges as the optimal choice. Similarly, the same order of ranking is obtained while fixing $\breve{q}$=4 and $\breve{q}$=5 respectively.

3.5 Comparative Examination

In this study, we integrated MULTIMOORA procedure with the proposed prioritized weighted aggregators for $\breve{q} = 3$ and $\breve{\gamma} = 2$. Then, we offer a comparison of our presented approach with established methods to assess the credibility and trustworthiness of the aggregators. The outcomes obtained using existing operators are notably comparable to our results, as depicted in Figure 1. Additionally, Table 7 highlights G_4 as the optimal choice when both existing and proposed aggregators are employed. This demonstrates the consistency between our ($\breve{q}$-RPFPWFA and $\breve{q}$-RPFPWFG) approach and the results of available aggregators, validating the method's reliability and authenticity.

4. CONCLUSIONS

Motivated by the characteristics and composition of $\breve{q}$-RPFs, we have presented a set of Frank aggregators in the context of $\breve{q}$-rung picture fuzzy systems. This study delves into the notion of $\breve{q}$-rung picture fuzzy prioritized weighted averaging and geometric operators within the framework of the MULTIMOORA procedure. We have devised two specific operators in this context: the $\breve{q}$-RPFPWFA aggregator and the -RPFPWFG aggregator. In decision-making problems, it is conceivable that the criteria and experts may have varying levels of importance. Assuming identical priority levels for both criteria and experts may not always be realistic. Hence, the $\breve{q}$-rung picture fuzzy prioritized weighted Frank aggregators come into play, allowing for the management of data that involves prioritization. In our study, we also conducted a comprehensive analysis based on the parameters $\breve{q}$ and $\breve{\gamma}$. Furthermore, it is crucial to complement aggregation procedure with multi-criteria methods. Ultimately, incorporating aggregation technique into MADM model MULTIMOORA not only enhances the decision-making process but also allows for a more effective representation of uncertainties.

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