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Research Paper

Modelling of Claim and Pricing of Motor Insurance Based on Bonus-Malus System Considering the Frequency and Severity of Claims

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Abstract

This article proposes new distributions for claim frequency and severity, specifically tailored for a bonus-malus system in automobile insurance. The mixed Poisson with weighted quasi Lindley distribution is recommended for modeling claim frequency, while the mixed exponential with weighted quasi Lindley distribution is suggested for modeling claim severity. To estimate insurance premiums, the Bayesian method is employed, incorporating both frequency and severity distributions. The study validates the proposed models using real data from an Australian insurance company, which includes 67,856 policies. The assessment of model adequacy indicates that the Poisson-weighted quasi Lindley distribution is a suitable fit for modeling claim frequency, while the exponential-weighted quasi Lindley distribution is appropriate for modeling claim severity. Overall, the results suggest that the proposed models offer optimal premium estimations, considering both claim frequency and severity, which can lead to fairer pricing and increased customer appeal during claim occurrences compared to conventional models.

Keywords

Bonus-Malus System, Motor Insurance, Claim Frequency, Claim Severity, Poisson Distribution

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1. INTRODUCTION

Automobile insurance has a crucial role in providing financial safety to motor owners against various risks, including accidents, third-party liabilities, and theft. The companies determine the premiums for motor insurance policies based on several factors, including the insured motor's characteristics, the driver's profile, and the policyholder's claims experience. One of the key mechanisms used by insurers to adjust premiums according to the policyholder's claims history is the bonus-malus system.

The bonus-malus system, also known as a no-claims bonus and claims malus, is extensively used in vehicle insurance to encourage insured individuals to drive safely and prevent accidents. This system rewards policyholders who do not file any claims during a policy year with a discount on their premiums, commonly called a bonus. Conversely, policyholders who do file claims may experience a rise in their premiums, referred to as a malus.

The usefulness of this system in encouraging safe driving behavior and reducing claims depends on several factors, including the frequency and severity of claims. Therefore, there is a need for accurate and reliable models to predict the impact of the bonus-malus system on claim frequency, claim severity, and overall insurance pricing.

The traditional bonus-malus system considers only the frequency of claims when calculating premiums. There are many articles related to the traditional bonus-malus system in insurance, which primarily consider the frequency of claims for premium calculation, including works by Tremblay (1992), Lemaire (1995), Walhin and Paris (1999), and Tzougas and Frangos (2014). However, this method leads to unfairness for all policyholders. For example, if a policyholder files one claim with a claim amount of 10000 Baht and another policyholder files one claim with a claim amount of 5000 Baht, both policyholders would have to pay the same premium in the following year. This results in an unfair situation for the second policyholder, who has a lower claim amount than the first policyholder. Several studies have proposed fairer premium models in the design of an optimal bonus-malus system (BMS), taking into account claim frequency and claim severity. These include Frangos and Vrontos (2001), Mert and Saykan (2005), Mahmoudvand and Hassani (2009), Ni et al. (2014), Tzougas et al. (2015), Ismail (2016), Deniz (2016), Deniz and Ojeda (2018), Tzougas et al. (2020), Jacob and Wu (2020), Moumeesri et al. (2020), and Ieosanurak et al. (2023). However, some papers have noted that existing models for claim frequency and claim severity are not yet suitable for car insurance data. This study proposes models for both claim frequency and claim severity, as well as pricing insurance premiums based on the bonus-malus system, considering both the frequency and severity of claims. This results in premiums are more appropriate and aligned with the history of claims from policyholders. Additionally, it benefits policyholders by allowing them to pay premiums that are fair and proportional to their individual risk profiles. Overall, this approach enhances competitiveness in the insurance market and contributes to the growth of the insurance business.

2. EXPERIMENTAL SECTION

The assumption that each claim from each individual is independent, and that the claim frequency and the amount of each claim from each individual are independent, can be described separately in the following subsections.

2.1 Claim Frequency

2.1.1 Mixing distributions

Consider $f(x;\theta)$ as the probability mass function (PMF) of the random variable X with a parameter θ. If the parameter θ vary, then θ is considered a random variable. Thus, there exists a conditional probability function $f(x|\theta)$ and the marginal distribution function of θ can be found by integrating the joint distribution function $f(x;\theta)$ over all possible values of the parameter θ. This can be expressed as:

$$f(x) = \int_{-\infty}^{\infty} f(x;\theta)d\theta = \int_{-\infty}^{\infty} f(x|\theta)g(\theta)d(\theta)$$

Here, $g(\theta)$ is the probability density function (PDF) of the random variable $\Theta = \theta$, and it is referred to as the prior distribution, representing the distribution function of the parameter θ.

In the context of car insurance, every policyholder has a risk parameter for the likelihood of an accident. This risk parameter is a random variable with values varying based on the driving history of each policyholder. The distribution of this random variable is called the prior distribution. In several works in the literature, it has been found that mixing a distribution with a prior distribution is more flexible and more suitable for modeling claim frequency than using a single distribution. We propose a frequency distribution of claims by mixing a Poisson distribution with a weighted quasi Lindley prior distribution as follows:

Let X represent the number of claims of a policyholder, where X is a random variable with a Poisson distribution. The PMF of the random variable X with a parameter λ is given by

$$f(x|\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}; \lambda > 0, x = 0, 1, 2, \ldots$$

The expected value and the variance of the random variable X are equal to the parameter λ, i.e., $E[X|\lambda] = Var[X|\lambda] = \lambda$.

Every policyholder faces the risk of experiencing an accident, which can be measured and naturally varies among individuals. This risk represents the average number of claims for each individual, denoted by λ. We assume that the parameter λ follows a weighted quasi Lindley distribution (Shanker et al., 2019) with parameters θ, β, and α. The PDF is given by

$$\begin{aligned} g(\lambda) =& \frac{\theta^\beta}{(\alpha+\beta)}\cdot\frac{\lambda^{\beta-1}}{\Gamma(\beta)}(\alpha+\theta\lambda)e^{-\theta\lambda}; \lambda > 0, \theta > 0, \\ & \beta > 0, \alpha > -1 \end{aligned}$$

The expected value of the random variable $\wedge$ is given by $E[\wedge] = \frac{\beta(\alpha+\beta+1)}{\theta(\alpha+\beta)}$.

We propose a claim frequency distribution function based on the mixed Poisson distribution with the weighted quasi Lindley distribution. This can be expressed as:

$$\begin{aligned} f(x) &= \int_0^\infty f(x|\lambda)g(\lambda)d\lambda \\ &= \int_0^\infty \frac{e^{-\lambda}\lambda^x}{x!}\cdot\frac{\theta^\beta}{(\alpha+\beta)}\frac{\lambda^{\beta-1}}{\Gamma(\beta)}(\alpha+\theta\lambda)e^{-\theta\lambda}d\lambda \\ &= \frac{\theta^\beta[\theta(\alpha+\beta+x)+\alpha]}{(\alpha+\beta)(1+\theta)^{x+\beta+1}}\frac{\Gamma(x+\beta)}{\Gamma(x+1)\Gamma(\beta)} \end{aligned}$$

The result obtained from the mixed Poisson distribution is called the Poisson-weighted quasi Lindley distribution (PWQLD). Its PMF is expressed as follows:

$$f(x;\theta,\beta,\alpha) = \frac{\theta^\beta[\theta(\alpha+\beta+x)+\alpha]}{(\alpha+\beta)(1+\theta)^{x+\beta+1}}\frac{\Gamma(x+\beta)}{\Gamma(x+1)\Gamma(\beta)}, \tag{1}$$

where $\theta > 0$, $\beta > 0$, $\alpha > -1$, and $x = 0, 1, 2, \ldots$.

It can be observed that when setting $\beta = 1$, the PWQLD reduces to the Poisson-quasi Lindley distribution (Grine and Zeghdoudi, 2017), whose PMF is represented as:

$$f(x;\theta,\alpha) = \frac{\theta[\theta(\alpha+x+1)+\alpha]}{(1+\alpha)(1+\theta)^{(x+2)}}.$$

2.1.2 Some Properties

In the upcoming section, we will explore the properties of the PWQLD. Here, we will derive its moment generating function (MGF), probability generating function (PGF), and the r-th moment about the origin.

Moment Generating Function

Let X be a random variable with the PWQLD, denoted as $X \sim PWQLD(\theta, \beta, \alpha)$. The MGF of X, denoted as $M_X(t)$, is given by

$$\begin{aligned} M_X(t) &= E(e^{tx}) \\ &= \int_0^\infty \sum_{x=0}^\infty e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}g(\lambda)d\lambda \\ &= \int_0^\infty \sum_{x=0}^\infty e^{tx}\frac{e^{-\lambda}\lambda^x}{x!}\cdot\frac{\theta^\beta}{(\alpha+\beta)}\frac{\lambda^{\beta-1}}{\Gamma(\beta)}(\alpha+\theta\lambda)e^{-\theta\lambda}d\lambda \end{aligned}$$

$$= \frac{\theta^{\beta}}{(\alpha+\beta)(\theta-e^{t}+1)^{\beta+1}}[\alpha(\theta-e^{t}+1)+\theta\beta].$$

Probability Generating Function

Let X be a random variable with the PWQLD, denoted as $X \sim \text{PWQLD}(\theta, \beta, \alpha)$. The PGF of X, denoted as $P_X(t)$, is given by

$$\begin{aligned} P_X(t) &= E(t^x) \\ &= \int_0^{\infty} \sum_{x=0}^{\infty} t^x \frac{e^{-\lambda}\lambda^x}{x!} g(\lambda) d\lambda \\ &= \int_0^{\infty} \sum_{x=0}^{\infty} t^x \frac{e^{-\lambda}\lambda^x}{x!} \cdot \frac{\theta^{\beta}}{(\alpha+\beta)} \frac{\lambda^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\lambda) e^{-\theta\lambda} \\ &\quad d\lambda \\ &= \frac{\theta^{\beta}}{(\alpha+\beta)(\theta-t+1)^{\beta}}[\alpha(\theta-t+1)+\theta\beta]. \end{aligned}$$

Moments

Let X be a random variable with the PWQLD, denoted as $X \sim \text{PWQLD}(\theta, \beta, \alpha)$. The r-th moment about the origin for X, denoted as $\mu'_{[r]}$, is given by

$$\mu'_{[r]} = \int_0^{\infty} \sum_{x=0}^{\infty} x^r \frac{e^{-\lambda}\lambda^x}{\Gamma(x+1)} \cdot \frac{\theta^{\beta}}{(\alpha+\beta)} \frac{\lambda^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\lambda) e^{-\theta\lambda} \, d\lambda$$

For the first moment about the origin ($r = 1$):

$$\begin{aligned} \mu'_{[1]} &= \int_0^{\infty} \sum_{x=0}^{\infty} x \frac{e^{-\lambda}\lambda^x}{\Gamma(x+1)} \cdot \frac{\theta^{\beta}}{(\alpha+\beta)} \frac{\lambda^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\lambda) e^{-\theta\lambda} \\ &\quad d\lambda \\ &= \int_0^{\infty} \lambda \frac{\theta^{\beta}}{(\alpha+\beta)} \frac{\lambda^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\lambda) e^{-\theta\lambda} \, d\lambda \\ &= \frac{\theta^{\beta}}{(\alpha+\beta)\Gamma(\beta)} \int_0^{\infty} \lambda^{\beta} e^{-\theta\lambda} (\alpha+\theta\lambda) \, d\lambda \\ &= \frac{\beta}{\theta(\alpha+\beta)}(\alpha+\beta+1). \end{aligned}$$

For the second moment about the origin ($r = 2$):

$$\begin{aligned} \mu'_{[2]} &= \int_0^{\infty} \sum_{x=0}^{\infty} x^2 \frac{e^{-\lambda}\lambda^x}{\Gamma(x+1)} \cdot \frac{\theta^{\beta}}{(\alpha+\beta)} \frac{\lambda^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\lambda) e^{-\theta\lambda} \\ &\quad d\lambda \\ &= \int_0^{\infty} (\lambda^2+\lambda) \frac{\theta^{\beta}}{(\alpha+\beta)} \frac{\lambda^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\lambda) e^{-\theta\lambda} \, d\lambda \\ &= \frac{\beta}{\theta^2(\alpha+\beta)}[\theta(\alpha+\beta+1)+(\beta+1)(\alpha+\beta+2)]. \end{aligned}$$

When substituting $r = 3$ and $r = 4$, the third and fourth moments about the origin are as follows:

$$\begin{aligned} \mu'_{[3]} &= \frac{\beta}{\theta^3(\alpha+\beta)}[\theta[\theta(\alpha+\beta+1)+3(\beta+1)(\alpha+\beta+ \\ &\quad 2)]+[(\beta+1)(\beta+2)(\alpha+\beta+3)]] \\ \mu'_{[4]} &= \frac{\beta}{\theta^4(\alpha+\beta)}[\theta[\theta^2+7\theta(\beta+1)(\alpha+\beta+2)+6(\beta+ \\ &\quad 1)+(\beta+2)(\alpha+\beta+3)]]+(\beta+1)[(\beta+2)(\beta+ \\ &\quad 3)(\alpha+\beta+4)+1] \end{aligned}$$

2.1.3 Bayesian Method

The Bayesian method has found applications in actuarial science and has been proven to be a valuable tool for addressing issues related to credibility theory in insurance pricing. The goal of using the Bayesian method in premium calculations is to assist in managing the posterior distribution in the process of accessing personal information, including the claim history or profiles of policyholders. The use of the Bayesian method is essential because it aids in estimating the risk and probability of future events related to the available data, including the claim history or profiles of policyholders. It helps improve the accuracy of risk assessments and premium calculations, making them more suitable and aligned with the existing data of policyholders.

Consider the sample data $x = (x_1, x_2, \dots, x_t)$, where t represents the specified time period, and $N = \Sigma_{i=1}^{t} x_i$ is the total frequency of claims for policyholders over the time period t. Here, x_i denotes the frequency of claims for policyholders in the i-th year, where $i = 1, 2, \dots, t$. The likelihood function is given by

$$\begin{aligned} L(\lambda; x_1, x_2, \dots, x_t) = f(x_1, x_2, \dots, x_t|\lambda) &= \prod_{i=1}^{t} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\ &\propto e^{-\lambda t}\lambda^{N}. \end{aligned}$$

Considering the prior distribution, we have

$$g(\lambda) = \frac{\theta^{\beta}}{\alpha+\beta} \frac{\lambda^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\lambda) e^{-\theta\lambda} \propto \lambda^{\beta-1}(\alpha+\theta\lambda) e^{-\theta\lambda}.$$

Using Bayes' theorem to find the posterior distribution function for the policyholders with claim history $x_1, x_2, \dots, x_t$, it is proportional to the product of the likelihood function and the prior distribution function. Thus, the posterior distribution function can be expressed as

$$\begin{aligned} g^*(\lambda|x_1, x_2, \dots, x_t) &\propto f(x_1, x_2, \dots, x_t|\lambda) g(\lambda) \\ &\propto e^{-\lambda t}\lambda^{N} \cdot \lambda^{\beta-1}(\alpha+\theta\lambda) e^{-\theta\lambda} \\ &\propto e^{-\lambda(t+\theta)}(\alpha+\theta\lambda)\lambda^{N+\beta-1}. \end{aligned}$$

The integral of $g^*(\lambda|x_1, x_2, \dots, x_t)$ over the entire range of λ is set to equal 1:

$$\int_0^{\infty} g^*(\lambda|x_1, x_2, ..., x_t), d\lambda = \int_0^{\infty} C e^{-\lambda(t+\theta)}(\alpha+\theta\lambda) \lambda^{N+\beta-1}, d\lambda = 1.$$

Solving the integral yields the constant C:

$$C = \frac{(t+\theta)^{N+\beta+1}}{\Gamma(N+\beta)\,[\theta(\alpha+\beta+N)+\alpha t]}.$$

This allows us to express the posterior distribution function of the PWQLD for the frequency components of claims as

$$g^*(\lambda|x_1, x_2, ..., x_t) = \frac{(t+\theta)^{N+\beta+1}}{\Gamma(N+\beta)\,[\theta(\alpha+\beta+N)+\alpha t]} [e^{-\lambda(t+\theta)}\lambda^{N+\beta-1}\alpha + e^{-\lambda(t+\theta)} \lambda^{N+\beta}\theta]$$

2.1.4 Insurance Premium Calculation

There are numerous principles for calculating insurance premiums, and in this article, we will focus on the net premium principle. This concept suggests that premiums should be calculated based on the average of losses (frequency and amount of claims) incurred. In this context, the premium signifies the average frequency or expected value of claims that policyholders are likely to file. Let $\hat{\lambda}_{t+1}$ represent the average frequency of claims for policyholders with claim histories $x_1, x_2, \ldots, x_t$. This value is equivalent to the expected value of the posterior distribution function of the PWQLD, expressed as follows:

$$\begin{aligned}\hat{\lambda}_{t+1} &= E[\lambda|x_1, x_2, ..., x_t]\\ &= \int_0^\infty \lambda\, g^*(\lambda|x_1, x_2, ..., x_t)\, d\lambda\\ &= \int_0^\infty \frac{\lambda(t+\theta)^{N+\beta+1}}{\Gamma(N+\beta)\,[\theta(\alpha+\beta+N)+\alpha t]}\\ &\quad \left[e^{-\lambda(t+\theta)}\lambda^{N+\beta-1}\alpha + e^{-\lambda(t+\theta)}\lambda^{N+\beta}\theta\right], d\lambda\\ &= \frac{(N+\beta)\,[\theta(\alpha+\beta+N+1)+\alpha t]}{(t+\theta)\,[\theta(\alpha+\beta+N)+\alpha t]}. \end{aligned} \tag{2}$$

Assuming that at $t = 0$, the initial premium is 100, the premium rate for the next period $(t+1)$ can be calculated using the following formula:

$$\begin{aligned}\text{Premium}_{t+1} &= \frac{\hat{\lambda}_{t+1}}{\hat{\lambda}_{0+1}} \times 100\%\\ &= \frac{(E[\lambda|x_1, x_2, \ldots, x_t])}{E[\lambda]} \times 100\%.\end{aligned}$$

Alternatively:

$$\text{Premium}_{t+1} = \frac{\frac{(N+\beta)[\theta(\alpha+\beta+N+1)+\alpha t]}{(t+\theta)[\theta(\alpha+\beta+N)+\alpha t]}}{\frac{\beta[\theta(\alpha+\beta+1)]}{\theta^2(\alpha+\beta)}} \times 100\%$$

$$= \frac{\theta(N+\beta)(\alpha+\beta)\,[\theta(\alpha+\beta+N+1)+\alpha t]}{\beta(t+\theta)(\alpha+\beta+1)\,[\theta(\alpha+\beta+N)+\alpha t]} \times 100\%. \tag{3}$$

where $\theta > 0, \beta > 0, \alpha > -1$, and $N = \Sigma_{(i=1)}^{t} x_i$.

2.2 Claim Severity

2.2.1 Mixing Distributions

The claim amounts or claim severity in insurance portfolios often exhibit a heavy-tailed distribution. Analysts typically focus on the right tail of the distribution. A mixed exponential distribution can be heavier-tailed compared to a single exponential distribution. This study proposes the combination of an exponential distribution with a prior distribution, specifically the weighted quasi Lindley distribution, to generate a model for the distribution of claim amounts in the insurance portfolio. The distribution of claim amounts for each policyholder is assumed to follow an exponential distribution, where Y represents a random variable denoting the claim amount for each policyholder. The PDF of Y with parameter γ is given by

$$f(y|\gamma) = \gamma e^{-\gamma y}; y > 0, \gamma > 0.$$

The expected value of the random variable Y is $E[Y|\gamma] = \frac{1}{\gamma}$.

Since the parameter γ varies for each policyholder, let $\ddot{\Gamma}$ be a random variable representing the parameter γ. Assume that the random variable $\ddot{\Gamma}$ follows a weighted quasi Lindley distribution with parameters τ, ϵ, and σ. The PDF of $\ddot{\Gamma}$ is given by

$$g(\gamma) = \frac{\tau^\epsilon}{\sigma+\epsilon}\frac{\gamma^{\epsilon-1}}{\Gamma(\epsilon)}(\sigma+\tau\gamma)e^{-\tau\gamma}; \gamma > 0, \tau, \epsilon > 0, \sigma > 0.$$

The expected value of the random variable $\ddot{\Gamma}$ is $E(\ddot{\Gamma}) = \frac{\epsilon(\sigma+\epsilon+1)}{\tau(\sigma+\epsilon)}$.

We propose a new continuous distribution function resulting from the mixture of the exponential distribution with a weighted quasi Lindley distribution, hereafter referred to as the exponential-weighted quasi Lindley distribution (EWQLD). This distribution represents an unconditional distribution of claim amounts and is expressed as follows:

$$\begin{aligned} f(y) &= \int_0^\infty f(y|\gamma)g(\gamma), d\gamma\\ &= \int_0^\infty \gamma e^{-\gamma y}\frac{\tau^\epsilon}{(\sigma+\epsilon)}\frac{\gamma^{\gamma-1}}{\Gamma(\epsilon)}(\sigma+\tau\gamma)e^{-\tau\gamma}, d\gamma\\ &= \frac{\tau^\epsilon\epsilon}{(\sigma+\epsilon)(y+\tau)^{\epsilon+2}}[\tau(\sigma+\epsilon+1)+y\sigma] \end{aligned} \tag{4}$$

where $\tau > 0, \epsilon > 0, \sigma > -1$ and $y > 0$.

2.2.2 Some Properties

Moment Generating Function

Let Y be a random variable with the EWQLD, denoted as $Y \sim \text{EWQLD}(\tau, \epsilon, \sigma)$. The MGF of Y, denoted as $M_Y(t)$, is given by

$$\begin{aligned} M_Y(t) &= E(e^{ty})\\ &= \int_0^\infty e^{ty}\frac{\tau^\epsilon\epsilon}{(\sigma+\epsilon)(y+\tau)^{\epsilon+2}}[\tau(\sigma+\epsilon+1)+y\sigma], dy\\ &= \frac{\tau^\epsilon\epsilon}{(\sigma+\epsilon)}\int_0^\infty e^{ty}\left[\frac{\tau(\sigma+\epsilon+1)+y\sigma}{(y+\tau)^{\epsilon+2}}\right], dy. \end{aligned}$$

The MGF of EWQLD cannot be expressed in closed form.

Probability Generating Function

Let Y be a random variable with the EWQLD, denoted as $Y \sim \text{EWQLD}(\tau, \epsilon, \sigma)$. The PGF of Y, denoted as $P_Y(t)$, is given by

$$\begin{aligned} P_Y(t) &= E(t^y) \\ &= \int_0^\infty t^y \frac{\tau^\epsilon \epsilon}{(\sigma+\epsilon)(y+\tau)^{\epsilon+2}}[\tau(\sigma+\epsilon+1)+y\sigma], \\ &\quad d\gamma \\ &= \frac{\tau^\epsilon \epsilon}{(\sigma+\epsilon)} \int_0^\infty \frac{t^y[\tau(\sigma+\epsilon+1)+y\sigma]}{(y+\tau)^{\epsilon+2}}, d\gamma. \end{aligned}$$

The PGF of the EWQLD cannot be expressed in closed form.

Moments

Let Y be a random variable with the EWQLD, denoted as $Y \sim \text{EWQLD}(\tau,\epsilon,\sigma)$. The r-th moment about the origin for Y, denoted as $\mu'_{[r]}$, is given by

$$\mu'_{[r]} = \int_0^\infty \int_0^\infty y^r \gamma e^{-\gamma y} dy \cdot \frac{\theta^\beta}{(\alpha+\beta)} \frac{\gamma^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\gamma) e^{-\theta\gamma} d\lambda.$$

For the first moment about the origin ($r = 1$):

$$\begin{aligned} \mu'_1 &= \int_0^\infty \int_0^\infty y\gamma e^{-\gamma y} dy \cdot \frac{\theta^\beta}{(\alpha+\beta)} \frac{\gamma^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\gamma) e^{-\theta\gamma} \\ &\quad d\gamma \\ &= \int_0^\infty \gamma^{-1} \frac{\theta^\beta}{(\alpha+\beta)} \frac{\gamma^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\gamma) e^{-\theta\gamma} d\gamma \\ &= \frac{\theta}{(\alpha+\beta)(\beta-1)}[\alpha+\beta-1]. \end{aligned}$$

For the second moment about the origin ($r = 2$):

$$\begin{aligned} \mu'_2 &= \int_0^\infty \int_0^\infty y^2 \gamma e^{-\gamma y} dy \\ &\quad \cdot \frac{\theta^\beta}{(\alpha+\beta)} \frac{\gamma^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\gamma) e^{-\theta\gamma} d\gamma \\ &= \int_0^\infty 2\gamma^{-2} \cdot \frac{\theta^\beta}{(\alpha+\beta)} \frac{\gamma^{\beta-1}}{\Gamma(\beta)} (\alpha+\theta\gamma) e^{-\theta\gamma} d\gamma \\ &= \frac{2\theta^2}{(\alpha+\beta)(\beta-1)(\beta-2)}[\alpha+\beta-2]. \end{aligned}$$

When substituting $r = 3$ and $r = 4$, the third and fourth moments about the origin are as follows:

$$\begin{aligned} \mu'_3 &= \frac{6\theta^3}{(\alpha+\beta)(\beta-1)(\beta-2)(\beta-3)}[\alpha+\beta-3] \\ \mu'_4 &= \frac{24\theta^4}{(\alpha+\beta)(\beta-1)(\beta-2)(\beta-3)(\beta-4)} \\ &\quad [\alpha+\beta-4]. \end{aligned}$$

2.2.3 Bayesian Method

Recall that the total number of claims made by the policyholders over a period of t years can be expressed as the sum of the number of claims made by each individual, denoted as $N = \Sigma_{i=1}^{t} x_i$. The amount of each claim is represented by y_j, where $j = 1, 2, \ldots, N$, or in vector form $y = (y_1, y_2, \ldots, y_N)$. Therefore, the total claim amount made by policyholders over the t-year period is represented by $S = \Sigma_{j=1}^{N} y_j$.

The likelihood function can be expressed as follows:

$$\begin{aligned} L(\gamma; y_1, y_2, \ldots, y_N) &= f(y_1, y_2, \ldots, y_N|\gamma) \\ &= \prod_{j=1}^{N} \gamma e^{-\gamma y_j} = \gamma^N e^{-\gamma \sum_{j=1}^{N} y_j} \\ &= \gamma^N e^{-\gamma S}. \end{aligned}$$

And the prior distribution can be expressed as:

$$g(\gamma) \propto^{(\epsilon-1)} (\sigma+\tau\gamma) e^{-\tau\gamma}.$$

Applying Bayes' theorem to find the posterior distribution function, which is proportional to the product of the likelihood function and the prior distribution function, we have

$$\begin{aligned} g^*(\gamma|y_1, y_2, \ldots, y_N) &\propto f(y_1, y_2, \ldots, y_N|\gamma) g(\gamma) \\ &\propto \gamma^N e^{-\gamma S} \gamma^{\epsilon-1} (\sigma+\tau\gamma) e^{-\tau\gamma} \\ &\propto (e^{-\gamma(S+\tau)} \gamma^{N+\epsilon-1} \sigma) + (e^{-\gamma(S+\tau)} \\ &\quad \gamma^{N+\epsilon} \tau). \end{aligned}$$

Considering the integral:

$$\begin{aligned} \int_0^\infty g^*(\gamma|y_1, y_2, \ldots, y_N) d\gamma &\propto \int_0^\infty (e^{-\gamma(S+\tau)} \gamma^{N+\epsilon-1} \sigma) + \\ &\quad (e^{-\gamma(S+\tau)} \gamma^{N+\epsilon} \tau)\, d\gamma. \end{aligned}$$

We find

$$\begin{aligned} \int_0^\infty g^*(\gamma|y_1, y_2, \ldots, y_N) d\gamma &= \int_0^\infty D(e^{-\gamma(S+\tau)} \gamma^{N+\epsilon-1} \sigma) + \\ &\quad (e^{-\gamma(S+\tau)} \gamma^{N+\epsilon} \tau)\, d\gamma = 1 \end{aligned}$$

This leads to:

$$D = \frac{(S+\tau)^{N+\epsilon+1}}{\Gamma(N+\epsilon)[\tau(\sigma+\epsilon+N)+\sigma S]}.$$

Thus, we can express the posterior distribution function of the EWQLD for the component of the claim severity as:

$$\begin{aligned} g^*(\gamma|y_1, y_2, \ldots, y_N) &= \frac{(S+\tau)^{N+\epsilon+1}}{\Gamma(N+\epsilon)[\tau(\sigma+\epsilon+N)+\sigma S]} \\ &\quad [(e^{-\gamma(S+\tau)} \gamma^{N+\epsilon-1} \sigma) + (e^{-\gamma(S+\tau)} \\ &\quad \gamma^{N+\epsilon} \tau)]. \end{aligned}$$

2.2.4 Insurance Premium Calculation

We employ the net premium principle for calculating premiums, which is typically calculated by taking the expected value of the claims. Therefore, the expected value of the posterior distribution function for the EWQLD is as follows:

$$\begin{aligned}\hat{\gamma}_{t+1} &= E[\gamma|y_1, y_2, ..., y_N] \\ &= \int_0^\infty \gamma \frac{(S+\tau)^{N+\epsilon+1}}{\Gamma(N+\epsilon)[\tau(\sigma+\epsilon+N)+\sigma S]} \\ &\quad \left[(e^{-\gamma(S+\tau)}\gamma^{N+\epsilon-1}\sigma)+(e^{-\gamma(S+\tau)}\gamma^{N+\epsilon}\tau)\right] d\gamma \\ &= \frac{(N+\epsilon)[\tau(\sigma+\epsilon+N+1)+\sigma S]}{(S+\tau)[\tau(\sigma+\epsilon+N)+\sigma S]}\end{aligned}$$

The posterior distribution illustrates an approximate distribution of the risk parameter, considering observed data. Specifically, it represents a conditional probability distribution of the risk parameter γ under observed data y. The primary objective of this article is to calculate the expected value of the observed data or claim values under the risk parameter, or in other words, the expected value of events in the future under past events (the risk parameter). However, since we do not have precise data regarding past events or that risk parameter, Bayesian method is used to determine the risk parameter value that aligns with the expected value of the posterior distribution function.

It is known that the expected value of an exponential random variable is $\frac{1}{\gamma}$, denoted as $E[Y|\gamma] = \frac{1}{\gamma}$. Since we do not know the true value of γ, we utilize Bayesian method, which depends on observed data, to determine the risk parameter (γ). The expected value of the posterior distribution function for the EWQLD can be written as

$$\hat{\gamma} = E[\gamma|y_1, y_2, ..., y_N].$$

Given $E[Y|\gamma] = \frac{1}{\gamma}$, the formula for $E[y_1, y_2, \ldots, y_N|\gamma] = \frac{1}{\hat{\gamma}}$ can be rewritten as:

$$E[y_1, y_2, ..., y_N|\gamma] = \frac{(S+\tau)[\tau(\sigma+\epsilon+N)+\sigma S]}{(N+\epsilon)[\tau(\sigma+\epsilon+N+1)+\sigma S]}. \quad (5)$$

The premiums for individual policyholders should directly reflect the frequency and amount of their claims. This means that policyholders with different claim values should affect their premiums differently, even if they have the same number of claim occurrences. To ensure fairness among policyholders, both the frequency and amount of claims should be considered together in calculating premiums. Therefore, the fair premium for all policyholders in the portfolio, obtained by multiplying Equation (2) and Equation (5), is expressed as:

$$\begin{aligned}\text{Premium}_{t+1} = &\frac{(N+\beta)[\theta(\alpha+\beta+N+1)+\alpha t]}{(t+\theta)[\theta(\alpha+\beta+N)+\alpha t]}\times \\ &\frac{(S+\tau)[\tau(\sigma+\epsilon+N)+\sigma S]}{(N+\epsilon)[\tau(\sigma+\epsilon+N+1)+\sigma S]}. \quad (6)\end{aligned}$$

Note that when the policyholder initiates the contract, the first premium payment occurs at $t = 0$, implying $N = 0$ and $S = 0$. Thus, the above formula can be simplified as shown in the following equation:

$$\text{Premium}_1 = \frac{\beta(\alpha+\beta+1)}{\theta(\alpha+\beta)} \times \frac{\tau(\sigma+\epsilon)}{\epsilon(\sigma+\epsilon+1)}. \quad (7)$$

3. RESULTS AND DISCUSSION

The data used in this study are from a one-year motor insurance dataset in 2004 or 2005 from an insurance company in Australia (De Jong and Heller, 2008). The dataset comprises a total of 67,856 policies. Among them, there are 4,333 policyholders who filed one claim, 271 policyholders who filed two claims, 18 policyholders who filed three claims, and 2 policyholders who filed four claims, as shown in Table 1.

3.1 Model Fitting

In this study, we utilize the PWQLD in Equation (1) to analyze the distribution of claim frequencies. The PWQLD parameters are estimated using the maximum likelihood estimation (MLE) method, resulting in estimates of $\hat{\theta} = 16.1921$, $\hat{\beta} = 1.1545$, and $\hat{\alpha} = 47.6080$. We then evaluate the fit of the PWQLD to the claim frequency data using a Chi-Square goodness of fit test, which yields a X^2 value of 1.0448. To further refine our analysis, we employ a randomized neighborhood search (RNS) technique to minimize the X^2 value. The RNS process is described in detail by Boonta et al. (2013). Using the RNS technique, we estimate the PWQLD parameters to be $\hat{\theta} = 15.9000$, $\hat{\beta} = 1.1406$, and $\hat{\alpha} = 67.1410$, with an adjusted X^2 value of 1.0366. These adjusted parameters provide a more precise representation of the data compared to the initial MLE estimates.

Additionally, we compare the expected frequency with other distributions, including the Poisson distribution, Poisson-one parameter Lindley distribution, and Poisson-two parameter Lindley distribution, as shown in Table 1.

In this article, we use a Chi-Square goodness of fit test to evaluate the suitability of the proposed claim frequency distribution and the Kolmogorov-Smirnov (K-S) goodness of fit test to assess the proposed claim severity distribution (Hogg et al., 1977). Additionally, we employ the Akaike information criterion (AIC) proposed by Akaike (1974) and the Bayesian information criterion (BIC) proposed by Stone (1979) as statistical criteria for model selection.

Table 1 displays the observed and expected frequencies for various models, including the Poisson, Poisson-one parameter Lindley (Sankaran, 1970), Poisson-two parameter Lindley (Shanker and Mishra, 2014), and Poisson-weighted quasi Lindley distribution. The goodness of fit is assessed using the X^2 statistic, with lower values indicating better fit. Notably, the Poisson-weighted quasi Lindley model shows the lowest X^2 value, suggesting superior fit compared to the other models. This indicates that the Poisson-weighted quasi Lindley model is the most suitable among those examined, as it closely matches observed and expected frequencies and has the lowest goodness-of-fit statistic. However, it should be noted that the

Table 1. Comparison of Observed and Expected Frequencies under Mixed Poisson Distributions

Number of Claims	Observed Frequency	Expected Frequency			
		Poisson	Poisson-one Parameter Lindley	Poisson-Two Parameter Lindley	Poisson-Weighted Quasi Lindley
0	63232	62905.86	63250.45	63233.58	63233.17
1	4333	4765.02	4293.96	4325.28	4326.51
2	271	180.47	290.57	278.74	277.73
3	18	4.56	19.61	17.29	17.43
4	2	0.09	1.32	1.04	1.08
5+	0	0.00	0.09	0.06	0.07
Total	67856	67856	67856	67856	67856
Estimated Parameters by MLE		$\hat{\theta}$= 0.0728	$\hat{\theta}$= 14.6238	$\hat{\theta}$ = 0.0987 $\hat{\alpha}$ = 18.5921	$\hat{\theta}$= 16.1921 $\hat{\beta}$= 1.1545 $\hat{\alpha}$ = 47.6080
X^2_{MLE}		177.9421	2.2507	1.2135	1.0448
Adjusted Estimated Parameters		$\hat{\theta}$ = 0.0758	$\hat{\theta}$ = 14.6166	$\hat{\theta}$ = 0.1121 $\hat{\alpha}$ = 8.2500	$\hat{\theta}$ = 15.9000 $\hat{\beta}$ = 1.1406 $\hat{\alpha}$ = 67.1410
X^2 Adjusted		168.3718	2.2495	1.1954	1.0366
AIC		36213.13	36102.76	36103.50	36105.38
BIC		36233.38	36123.01	36144.00	36166.13

Poisson-weighted quasi Lindley model tends to show no difference in the AIC and BIC values compared to other models. Therefore, AIC and BIC values may not be useful for selecting the model for the claim frequency distributions.

To compare the adequacy of claim severity distribution, this study employed mixed distributions derived from mixing exponential distribution with one-parameter Lindley, two-parameter Lindley, and weighted quasi Lindley distribution. The PDF and cumulative distribution function (CDF) of these distributions are derived and are presented in Table 2.

Table 3 presents the parameters and statistical values for various claim severity distributions, including exponential, exponential-one parameter Lindley, exponential-two parameter Lindley, and exponential-weighted quasi Lindley distribution in Equation (4). The K-S statistic is provided for each distribution model, with lower values indicating a better fit of the model to the data. The exponential-weighted quasi Lindley distribution shows the lowest K-S statistic value, suggesting superior fit compared to other models. Additionally, AIC and BIC values are presented, with lower values indicating better model fit while considering model complexity. The exponential-weighted quasi Lindley distribution exhibits the lowest AIC and BIC values among the models considered, indicating its potential superiority in balancing goodness of fit. Overall, the results suggest that the exponential-weighted quasi Lindley distribution may be the most suitable model for representing the claim severity distributions.

3.2 Insurance Premium Pricing

For pricing premiums, we consider each subsection separately: one for the claim frequency component and another for the claim frequency and severity components, as outlined in the following subsections.

3.2.1 Claim Frequency Component

Table 4 provides bonus-malus premiums or premium rates calculated from Equation (3) based on the exponential-weighted quasi Lindley distribution for the frequency component, categorized by the number of years since policy inception and the number of claims made within each year. The premiums appear to vary based on both the time since policy inception and the number of claims within each year. Generally, as the number of claims increases within a given year or as time progresses since policy inception, the premiums tend to increase. This aligns with the principle of bonus-malus systems, where policyholders are rewarded with lower premiums for a history of few or no claims (bonus), while premiums increase for those with a higher frequency of claims (malus). The premiums presented in the table may reflect the insurer's risk assessment based on claim frequency patterns observed over time. Insurers typically adjust premiums to account for the likelihood of future claims, and the data in the table likely inform these adjustments. Table 4 provides insight into how bonus-malus premiums are calculated based on claim frequency data and illustrates how insurers incorporate risk assessment into premium pricing strategies.

From Table 4, it can be observed that policyholders with no claims in the first year will receive a bonus or premium discount of 6% of the initial premium. However, if a policyholder makes one claim in the first year, they will have to pay an additional 76.39% of the initial premium. It is noticeable that the premium decreases when policyholders do not make any claims, and it increases when claims are made.

Table 2. The PDF and CDF of Mixed Exponential Distribution with Various Prior Distributions

Exponential	PDF :	$f(y;\lambda) = \lambda e^{-\lambda y}, y > 0, \lambda > 0$
	CDF :	$F(y;\lambda) = 1 - e^{-\lambda y}$
Exponential-One Parameter Lindley	PDF :	$f(y;\delta) = \frac{\delta^2}{\delta+1} \cdot \frac{y+\delta+2}{(y+\delta)^3}, y > 0, \delta > 0$
	CDF :	$F(y;\delta) = \frac{(\delta+1)y^2+(\delta+2)\delta y}{(\delta+1)(y+\delta)^2}$
Exponential-Two Parameter Lindley	PDF:	$f(y;\alpha,\delta) = \frac{\delta^2}{(\alpha\delta+1)} \cdot \frac{\alpha y+\alpha\delta+2}{(y+\delta)^3}, y > 0, \alpha, \delta > 0$
	CDF :	$F(y;\alpha,\delta) = \frac{(\alpha\delta+1)y^2+(\alpha\delta+2)\delta y}{(\alpha\delta+1)(y+\delta)^2}$
Exponential-Weighted Quasi Lindley	PDF :	$f(y;\tau,\epsilon,\sigma) = \frac{\tau^\epsilon \epsilon}{(\sigma+\epsilon)(y+\tau)^{\epsilon+2}}[\tau(\sigma+\epsilon+1)+y\sigma]$, $y > 0, \tau > 0, \epsilon > 0, \sigma > -1$
	CDF :	Not closed form

Table 3. Comparison of Parameters and Statistic Values for Claim Severity Distributions

	Claim Severity Distribution			
	Exponential	Exponential-One Parameter Lindley	Exponential-Two Parameter Lindley	Exponential-Weighted Quasi Lindley
Estimated Parameters by MLE	$\hat{\lambda} = 0.000533$	$\hat{\delta} = 977.0534$	$\hat{\alpha} = 16.8750$ $\hat{\delta} = 976.2500$	$\hat{\tau} = 5765.5199$ $\hat{\epsilon} = 4.0418$ $\hat{\sigma} = 1288.8348$
K-S statistic	0.1034	0.0880	0.0880	0.0416
AIC	78956.80	79416.58	79418.58	78708.33
BIC	78971.68	79431.46	79448.33	78752.96

Table 4. Premium Rate Based on the Poisson-Weighted Quasi Lindley Distribution for the Frequency Component

t	Number of Claims				
	0	1	2	3	4
0	100.00				
1	94.00	176.39	258.74	341.07	423.37
2	88.69	166.41	244.11	321.79	399.45
3	83.94	157.50	231.05	304.57	378.08
4	79.67	149.50	219.31	289.11	358.88
5	75.82	142.27	208.71	275.13	341.54
6	72.32	135.71	199.08	262.45	325.80
7	69.13	129.72	190.31	250.88	311.44

Table 5. Premium Rate Based on the Poisson-One Parameter Lindley Distribution for the Frequency Component

t	Number of claims				
	0	1	2	3	4
0	100.00				
1	93.26	185.91	278.07	369.79	461.15
2	87.36	174.22	260.65	346.72	432.47
3	82.17	163.90	245.28	326.34	407.13
4	77.55	154.74	231.61	308.21	384.57
5	73.42	146.53	219.37	291.97	364.36
6	69.71	139.15	208.36	277.35	346.16
7	66.36	132.48	198.39	264.12	329.68

Moreover, this article has compared bonus-malus premiums based on the Poisson-one parameter Lindley distribution (Moumeesri et al., 2020). As shown in Table 5, policyholders with no claims in the first year will receive a bonus or premium discount of 6.74% of the initial premium. However, if a policyholder makes one claim in the first year, they will have to pay an additional 85.91% of the initial premium.

The bonus-malus premiums calculated using the Poisson-weighted quasi Lindley distribution result in lower premium discounts compared to the Poisson-one parameter Lindley distribution. When there are no claims, the premium discounts are lower with the Poisson-weighted quasi Lindley distribution. On the other hand, when claims occur, the premium increases are lower with the Poisson-weighted quasi Lindley distribution. This implies that the Poisson-weighted quasi Lindley distribution provides more leniency for policyholders with claims. In the context of insurance companies, the Poisson-weighted quasi Lindley distribution can attract more customers, and policyholders are less likely to switch contracts to other companies when claims occur, compared to the Poisson-one parameter Lindley distribution.

3.2.2 Claim Frequency and Claim Severity Components

Tables 6-8 illustrate the bonus-malus premiums based on the frequency of claims and the total severity of those claims. The rows represent different policy periods (t), while the columns indicate the number of claims made during each period. The values in the table denote the corresponding insurance premium amounts in Australian Dollar (AUD). We consider scenarios where policyholders make claims with a total claim amount of S = 300, 500, and 1,000 AUD.

Tables 6-8 depict the premiums that policyholders must pay, calculated from Equation (6), considering both the number of claims and the severity of the claims. These calculations are based on a 7-year policy term, utilizing the Poisson-weighted quasi Lindley distribution for the frequency component of claims and the exponential-weighted quasi Lindley distribution for the severity component of claims. The initial insurance premium for all policyholders, calculated from Equation (7), is set at 103.75 AUD, and it decreases when no claims are made. If a policyholder makes one claim in the first year with a total amount of 300 AUD, the premium increases to 154.34 AUD, as shown in Table 6.

In the second year, if a policyholder makes one claim with a total value of 200 AUD, the premium increases to 184.13 AUD, as shown in Table 7. This indicates that after 2 years, the policyholder has made a total of 2 claims with a combined amount of 500 AUD.

In the third year, if the policyholder does not make any claims, the premium decreases to 174.28 AUD, as depicted in Table 7. This implies that after 3 years, the policyholder has made a total of 2 claims with a combined amount of 500 AUD.

In the fourth year, if the policyholder makes 2 claims with values of 100 AUD and 400 AUD, respectively, the premium increases to 219.62 AUD, as shown in Table 8. This indicates that after 4 years, the policyholder has made a total of 4 claims with a combined amount of 1,000 AUD.

Furthermore, from Tables 6-8, with the total claim amount remaining fixed, it becomes evident that as the number of claims increases, the premium also increases.

Table 6. Bonus-Malus Premiums Based on Claims Frequency and Severity with S = 300 AUD

t	Number of Claims				
	0	1	2	3	4
0	103.75				
1	97.53	**154.34**	188.93	213.68	232.25
2	92.01	145.61	178.25	201.60	219.13
3	87.08	137.82	168.71	190.81	207.41
4	82.66	130.81	160.14	181.12	196.88
5	78.66	124.49	152.40	172.37	187.37
6	75.03	118.75	145.37	164.42	178.73
7	71.72	113.51	138.96	157.17	170.85

Table 7. Bonus-Malus Premiums Based on Claims Frequency and Severity with S = 500 AUD

t	Number of Claims				
	0	1	2	3	4
0	103.75				
1	97.53	159.43	195.16	220.73	239.92
2	92.01	150.42	**184.13**	208.25	226.36
3	87.08	142.37	**174.28**	197.11	214.25
4	82.66	135.13	165.42	187.10	203.38
5	78.66	128.60	157.43	178.06	193.55
6	75.03	122.66	150.17	169.85	184.63
7	71.72	117.26	143.54	162.36	176.49

Table 8. Bonus-Malus Premiums Based on Claims Frequency and Severity with S = 1000 AUD

t	Number of Claims				
	0	1	2	3	4
0	103.75				
1	97.53	172.17	210.75	238.35	259.08
2	92.01	162.43	198.83	224.88	244.44
3	87.08	153.73	188.19	212.85	231.36
4	82.66	145.92	178.63	202.04	**219.62**
5	78.66	138.87	170.00	192.28	209.00
6	75.03	132.46	162.16	183.41	199.37
7	71.72	126.62	155.01	175.33	190.58

4. CONCLUSIONS

The model adequacy assessment indicates that the Poisson-weighted quasi Lindley distribution is a suitable fit for modeling claim frequencies, while the exponential-weighted quasi Lindley distribution is appropriate for modeling claim amounts. The proposed models, using the mixed Poisson with weighted quasi Lindley distribution for claim frequencies and the mixed exponential with weighted quasi Lindley distribution for claim amounts, provide optimal premium estimations considering both claim frequencies and amounts. This leads to fairer pricing and increased customer appeal during claim occurrences compared to conventional models. Further research can explore the applicability of the proposed models in different insurance contexts beyond motor insurance, such as health insurance or property insurance, to assess their effectiveness and accuracy. The proposed models can be further refined and enhanced by incorporating additional factors that may influence claim frequencies and amounts, such as driver characteristics, vehicle type, or geographical location.

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REFERENCES

Akaike, H. (1974). A New Look at the Statistical Model Identification. *IEEE Transactions on Automatic Control*, **19**(6); 716–723

Boonta, S., A. Sattayatham, and P. Sattayatham (2013). Estimation of Weibull Parameters Using a Randomized Neighborhood Search for the Severity of Fire Accidents. *Journal of Mathematics and Statistics*, **9**(1); 12–17

De Jong, P. and G. Z. Heller (2008). *Generalized Linear Models for Insurance Data*. Cambridge University Press, Cambridge

Deniz, E. (2016). Bivariate Credibility Bonus-Malus Premiums Distinguishing between Two Types of Claims. *Insurance: Mathematics and Economics*, **70**; 117–124

Deniz, E. and E. Ojeda (2018). Multivariate Credibility in Bonus-Malus Systems Distinguishing between Different Types of Claims. *Risks*, **6**(2); 34

Frangos, N. E. and S. D. Vrontos (2001). Design of Optimal Bonus-Malus Systems with a Frequency and a Severity Component on an Individual Basis in Automobile Insurance. *Astin Bulletin*, **31**; 1–22

Grine, R. and H. Zeghdoudi (2017). On Poisson Quasi-Lindley Distribution and its Applications. *Journal of Modern Applied Statistical Methods*, **16**(2); 403–417

Hogg, R. V., E. A. Tanis, and D. L. Zimmerman (1977). *Probability and Statistical Inference*. Macmillan, New York

Ieosanurak, W., B. Khomkham, and A. Moumeesri (2023). Claim Modeling and Insurance Premium Pricing under a Bonus-Malus System in Motor Insurance. *International Journal of Applied Mathematics and Computer Science*, **33**(4); 637–650

Ismail, E. (2016). Bayesian Approach for Bonus-Malus Systems with Gamma Distributed Claim Severities in Vehicles Insurance. *British Journal of Economics, Management & Trade*, **14**(1); 1–9

Jacob, A. and Z. Wu (2020). An Alternative Pricing System through Bayesian Estimate and Method of Moments in Bonus-Malus Framework for the Ghanaian Auto Insurance Market. *Journal of Risk and Financial Management*, **13**(7); 143–157

Lemaire, J. (1995). *Bonus-Malus Systems in Automobile Insurance*. Springer

Mahmoudvand, R. and H. Hassani (2009). Generalized Bonus-Malus Systems with a Frequency and a Severity Component on an Individual Basis in Automobile Insurance. *ASTIN Bulletin: The Journal of the IAA*, **39**(1); 307–315

Mert, M. and Y. Saykan (2005). On a Bonus-Malus System where the Claim Frequency Distribution is Geometric and the Claim Severity Distribution is Pareto. *Hacettepe Journal of Mathematics and Statistics*, **34**; 75–81

Moumeesri, A., W. Klongdee, and T. Pongsart (2020). Bayesian Bonus-Malus Premium with Poisson-Lindley Distributed Claim Frequency and Lognormal-Gamma Distributed Claim Severity in Automobile Insurance. *WSEAS Transactions on Mathematics*, **19**; 443–451

Ni, W., C. Corina, and P. Athanasios (2014). Bonus-Malus Systems with Weibull Distributed Claim Severities. *Annals of Actuarial Science*, **8**(2); 217–233

Sankaran, M. (1970). The Discrete Poisson–Lindley Distribution. *Biometrics*, **26**(1); 145–149

Shanker, R. and A. Mishra (2014). A Two-Parameter Poisson–Lindley Distribution. *International Journal of Statistics and Systems*, **9**(1); 79–85

Shanker, R., K. K. Shukla, and T. Leonida (2019). Weighted Quasi Lindley Distribution with Properties and Applications. *International Journal of Statistics and Applications*, **9**(1); 8–20

Stone, M. (1979). Comments on Model Selection Criteria of Akaike and Schwarz. *Journal of the Royal Statistical Society. Series B (Methodological)*, **41**(2); 276–278

Tremblay, L. (1992). Using the Poisson Inverse Gaussian in Bonus-Malus Systems. *ASTIN Bulletin: The Journal of the IAA*, **22**(1); 97–106

Tzougas, G. and N. Frangos (2014). The Design of an Optimal Bonus-Malus System Based on the Sichel Distribution. In *Modern Problems in Insurance Mathematics*. Springer, pages 239–260

Tzougas, G., S. D. Vrontos, and N. E. Frangos (2015). Optimal Bonus-Malus Systems Using Finite Mixture Models. *ASTIN Bulletin: The Journal of the IAA*, **44**(2); 417–444

Tzougas, G., W. H. Yik, and M. W. Mustaqeem (2020). Insurance Ratemaking Using the Exponential-Lognormal Regression Model. *Annals of Actuarial Science*, **14**(1); 42–71

Walhin, J.-F. and J. Paris (1999). Using Mixed Poisson Processes in Connection with Bonus-Malus Systems. *ASTIN Bulletin: The Journal of the IAA*, **29**(1); 81–99

