Science and Technology Indonesia
e-ISSN:2580-4391 *p*-ISSN:2580-4405
Vol. 9, No. 4, October 2024

Research Paper

Some Coefficient Problems for Subclasses of Holomorphic Functions in Complex Order Associating Sălăgeăn q-Differential Operator

Tseu Suet Yie[1], Aini Janteng[2*], Muhammad Abbas[3]

[1]*Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, Sabah, 88450, Malaysia*
[2]*Mathematics and Statistics Applications Research Group, Faculty of Science and Natural Resources, Universiti Malaysia Sabah, Kota Kinabalu, Sabah, 88450, Malaysia*
[3]*Department of Mathematics, Abdul Wali Khan University Mardan, Mardan, 23200, Pakistan*
*Corresponding author: aini_jg@ums.edu.my

Abstract
A function with complex values and at every point of the specific domain contains a derivative is commonly known as analytic functions which is also referred as holomorphic functions. We begin by interpreting A as the class for all holomorphic functions $L(\xi)$ with a Taylor series expansion written in the form $L(\xi) = \xi + \Sigma_{i=2}^{\infty} x_i \xi^i$ where $x_i \in \mathbb{C}$ and $\xi \in D$. D is the open unit disk where $D = \{\xi : \xi \in \mathbb{C}, |\xi| < 1\}$. Furthermore, we suggest the subclass of A that is univalent in D represent as S. It is commonly known that the main subclasses of class S are the class of starlike function and the class of convex function. To develop and analyze the Fekete-Szegö problems, the theory of geometric function contributes significantly to this. Moreover, the frequent use of q-calculus as a general direction of research among mathematicians has caught our attention. In this research, we attained the initial coefficients, x_2 and x_3, and the upper bound for the functional $|x_3 - vx_2^2|$ of functions L in the two new subclasses that are introduced by involving the Sălăgeăn q-differential operator, $M_q^\eta L(\xi)$ and the definition of subordination.

Keywords
Analytic (or Holomorphic) Functions, Univalent Functions, q-Calculus, q-Derivative (or q-Differential) Operator, Fekete-Szegö Functional, Subordination

Received: 7 May 2024, Accepted: 4 September 2024
https://doi.org/10.26554/sti.2024.9.4.981-988

1. INTRODUCTION

As to get down along the study, we express A as the class for all holomorphic functions $L(\xi)$ in the open unit disk $D = \{\xi : \xi \in \mathbb{C}, |\xi| < 1\}$ and present as in the equation below:

$$L(\xi) = \xi + \Sigma_{i=2}^{\infty} x_1 \xi^i \tag{1}$$

Al-Shbeil et al. (2023) declared that (1) is also called the Taylor's series expansion of L. We also suggest S represent as the subclass of class A that is univalent in D. Kowalczyk and Lecko (2014) stated that it is commonly known in fact the class of starlike function and the class of convex function classified as the main subclasses of class S.

Khan et al. (2017) interpret the subordination as $L \prec h$ in D or $L(\xi) \prec h(\xi)$ with the condition of L and h are in A and $\xi \in D$.

However, Khan et al. (2022) declared that the interpretation of subordination remains valid if a Schwarz function $y(\xi)$ exists in the condition of it is holomorphic in D, $y(0) = 0$ and $|y(\xi)| < 1$ for all $\xi \in D$. For example, $L(\xi) = h(y(\xi))$ where $\xi \in D$.

In the late 19[th] and early 20[th] centuries, the subject of geometric function theory has turned into mathematicians' research inclination as to investigate the mapping between complex analytic functions and their geometric characteristics (Oros, 2023). Fekete-Szegö problem is found by Fekete and Szeö (1933) and is a long-established geometric function theory that emphasize on the functional $|x_{i+1} - x_i|$, where x_i express as the i^{th} coefficient of the expansion of the power series of a holomorphic function. Furthermore, Fekete-Szegö inequality yields a specific inequality for coefficients, but the Fekete-Szegö functional designates a more comprehensive class of holomorphic functions on the basis of quasi-subordination principles (Srivastava et al., 2018).

We are aware about the topics of q-calculus have turn into a popular direction of research for the mathematicians. Jackson (1909, 1910) was the first generation of researcher that dived into the topics of q-integral and q-derivative more thoroughly.

The q-derivative operator is interpreted as

$$D_qL(\xi) = \begin{cases} \frac{L(q\xi)-L(\xi)}{(q-1)\xi} & \text{for } \xi \neq 0, 0 < q < 1 \\ L'(0) & \text{for } \xi = 0 \end{cases} \quad (2)$$

for functions L which are differentiable at $\xi = 0$ (Ramachandran et al., 2017). Then, $D_q^\eta L$ is further defined as

$$D_q^\eta L(\xi) = \xi D_q(D_q^{\eta-1}L\xi)$$

for $\eta = 1, 2, 3, \ldots,$ where $D_q^0 = L$ (Akca et al., 2019).

Utilizing the use of q-derivative operator, the classes, $S^*_{q,\eta}(\alpha)$ and $C_{q,\eta}(\alpha)$ are established and can be written as below

$$S^*_{(q,\eta)}(\alpha) = \left\{L \in A : \text{Re}\left(\frac{\xi D_q(M_q^\eta L(\xi))}{M_q^\eta L(\xi)}\right) > \alpha, \xi \in D\right\}$$

and

$$C_{(q,\eta)}(\alpha) = \left\{L \in A : \text{Re}\left(1 + \frac{\xi q D_q(D_q(M_q^\eta L(\xi)))}{D_q M_q^\eta L(\xi)}\right) > \alpha, \xi \in D\right\}$$

where $0 \leq \alpha < 1$, $0 < q < 1$ and $\eta \in N$ (Alsoboh and Darus, 2019).

There are numerous researchers that have done studies related to establishing subclasses of holomorphic functions by utilizing the interpretation of subordination. To be precise, some researchers have procured the coefficients, $|x_2|$ and $|x_3|$, and the upper bound for the Fekete-Szegö functional, $|x_3 - vx_2^2|$, of functions L. Apart from that, there are several researchers introduced multiple new subclasses involving q-derivative(or q-differential) operator (Andrei and Cuas, 2021; Amini et al., 2022; Ayinla and Opoola, 2019; Ahmad et al., 2021; Hussain et al., 2018; Hern et al., 2022; Hussain et al., 2017; Hu et al., 2022; ?; Naeem et al., 2019; Shaba et al., 2023; Srivastava et al., 2021; Saliu et al., 2020; Yie and Janteng, 2024; Janteng et al., 2020; Zainab et al., 2021). We generalized the ideas from these researchers, and we intend to acquire the initial coefficients along with the upper bound for the Fekete-Szegö functional of functions L for the two new subclasses involving the Sălăgeăn q-differential operator and the interpretation of subordination.

Alsoboh and Darus (2019)'s concept is our main reference where they established $M_q^\eta L(\xi)$ as the q-differential operator of a function L in the form of (1):

$$M_q^0 L(\xi) = L(\xi),$$

$$M_q^1 L(\xi) = \xi D_q L(\xi) = \xi + \sum_{i=2}^{\infty} [i]_q x_i \xi^i,$$

and

$$M_q^\eta L(\xi) = \xi D_q\left(M_q^{\eta-1}L(\xi)\right) = \xi + \sum_{i=2}^{\infty} [i]_q^\eta x_i \xi^i, \quad (3)$$

with $[i]_q = \frac{1-q^i}{1-q}$.

Let τ represent as the class of all functions φ that is holomorphic and univalent in D. The definitions of the new subclasses, $J^\eta_{q,\kappa}(\varphi)$ and $W^\eta_{q,k}(\varphi)$, are correspondingly mentioned below:

Definition 1 A function $L \in A$ belongs to the class $J^\eta_{q,\kappa}(\varphi)$ if the condition of subordination mentioned below remains valid

$$1 + \frac{1}{\kappa}\left(D_q\left(M_q^\eta L(\xi)\right) - 1\right) \prec \varphi(\xi),$$

$$\kappa \in \mathbb{C} \setminus \{0\},\ 0 < q < 1,\ \eta \in \mathbb{N},\ \varphi \in \tau,\ \xi \in D.$$

Definition 2 A function $L \in A$ belongs to the class $W^\eta_{q,\kappa}(\varphi)$ if the condition of subordination mentioned below remains valid

$$1 + \frac{1}{\kappa}\left((1-\delta)\frac{\xi D_q(M_q^\eta L(\xi))}{M_q^\eta L(\xi)} + \delta\left(1 + \frac{q\xi D_q(D_q M_q^\eta L(\xi))}{D_q(M_q^\eta L(\xi))}\right) - 1\right) \prec \varphi(\xi),$$

$$\kappa \in \mathbb{C} \setminus \{0\}, 0 < q < 1, \eta \in \mathbb{N}, 0 \leq \delta \leq 1, \varphi \in \tau, \xi \in D.$$

2. PRELIMINARIES RESULT

The preliminary results are crucial to manifest our key findings.

Lemma 1 (Ma and Minda, 1992) If $t(\xi) = 1 + t_1\xi + t_2\xi^2 + \cdots$ is a function with positive real part in D and $\zeta \in \mathbb{C}$, then

$$|t_2 - \zeta t_1^2| \leq 2\max\{1, |2\zeta - 1|\}.$$

The outcome is sharp for the functions particularly by

$$t(\xi) = \frac{1+\xi^2}{1-\xi^2}$$

and

$$t(\xi) = \frac{1+\xi}{1-\xi}.$$

Lemma 2 (Ma and Minda, 1992) If $t(\xi) = 1 + t_1\xi + t_2\xi^2 + \cdots$ is a function with positive real part in D, then

$$|t_2 - \zeta t_1^2| \leq \begin{cases} -2(2\zeta + 1), & \text{if } \zeta \leq 0, \\ 2, & \text{if } 0 \leq \zeta \leq 1, \\ 2(2\zeta - 1), & \text{if } \zeta \geq 1. \end{cases}$$

When $\zeta < 0$ or $\zeta > 1$, the equality remains valid if $t(\xi) = \frac{1+\xi}{1-\xi}$ or a rotation of itself. If $0 < \zeta < 1$, then the equality remains valid if $t(\xi) = \frac{1+\xi^2}{1-\xi^2}$ or a rotation of itself. If $\zeta = 0$, the equality remains valid if

$$t(\xi) = \frac{1}{2}\left((1+\vartheta)\frac{1+\xi}{1-\xi} + (1-\vartheta)\frac{1-\xi}{1+\xi}\right), \quad 0 \leq \vartheta \leq 1$$

or a rotation of itself. If $\zeta = 1$, the equality remains valid if $t(\xi)$ is the correlative with one of the functions in such a way that the equality remains valid when $\zeta = 0$. The upper bound mentioned in Lemma 2 is clear and it is possibly better when $0 < \zeta < 1$,

$$|t_2 - \zeta t_1^2| + \zeta |t_1|^2 \leq 2, \quad 0 \leq \zeta \leq \frac{1}{2},$$

and

$$|t_2 - \zeta t_1^2| + (1-\zeta)|t_1|^2 \leq 2, \quad \frac{1}{2} \leq \zeta \leq 1.$$

3. MAIN RESULTS

To declare the main results, we establish some theorems for the new subclasses.

Theorem 1 Let $\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \cdots$ with $B_1 \neq 0$, and let L, as interpreted by (1), belongs to the class $J^{\eta}_{q,\kappa}(\varphi)$ and $\nu \in \mathbb{C}$, then

$$|x_3 - \nu x_2^2| \leq \frac{|\kappa B_1|}{[3]_q^{\eta+1}} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{\nu\kappa [3]_q^{\eta+1} B_1}{[2]_q^{2\eta+2}}\right|\right\}. \quad (4)$$

Proof. If $L \in J^{\eta}_{q,\kappa}(\varphi)$, it results in the existence of a Schwarz function $y(\xi)$ where $y(0) = 0$ and $|y(\xi)| < 1$ in D such that

$$1 + \frac{1}{\kappa}\left(D_q\left(M_q^{\eta} L(\xi)\right) - 1\right) = \varphi(y(\xi)). \quad (5)$$

The function $t(\xi)$ is expressed as

$$t(\xi) = \frac{1 + y(\xi)}{1 - y(\xi)} = 1 + t_1\xi + t_2\xi^2 + \cdots \quad (6)$$

where $\mathrm{Re}(t(\xi)) > 0$ and $t(0) = 1$ with $y(\xi)$ as the Schwarz function. Let

$$h(\xi) = 1 + \frac{1}{\kappa}\left(D_q\left(M_q^{\eta} L(\xi)\right) - 1\right) = 1 + d_1\xi + d_2\xi^2 + \cdots. \quad (7)$$

From (5), (6), and (7), we get that

$$h(\xi) = \varphi\left(\frac{t(\xi) - 1}{t(\xi) + 1}\right). \quad (8)$$

By (6), we interpret $y(\xi)$ in terms of $t(\xi)$, and we have

$$y(\xi) = \frac{t(\xi) - 1}{t(\xi) + 1} = \frac{1}{2}\left(t_1\xi + \left(t_2 - \frac{t_1^2}{2}\right)\xi^2 + \left(t_3 + \frac{t_1^3}{4} - t_1 t_2\right)\xi^3 + \cdots\right). \quad (9)$$

From $\varphi(\xi)$, (8), and (9), we obtain that

$$h(\xi) = 1 + \frac{1}{2}B_1 t_1 \xi + \left(\frac{1}{2}B_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}B_2 t_1^2\right)\xi^2 + \cdots \quad (10)$$

Now, by implementing (1), (2), and (3), we obtain that

$$h(\xi) = 1 + \frac{1}{\kappa}\left(D_q\left(M_q^{\eta} L(\xi)\right) - 1\right) = 1 + \frac{[2]_q^{\eta+1} x_2 \xi}{\kappa} + \frac{[3]_q^{\eta+1} x_3 \xi^2}{\kappa} + \cdots. \quad (11)$$

By linking (10) and (11), we get that

$$d_1 = \frac{[2]_q^{\eta+1} x_2}{\kappa} = \frac{1}{2}B_1 t_1,$$

and

$$d_2 = \frac{[3]_q^{\eta+1} x_3}{\kappa} = \frac{1}{2}B_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}B_2 t_1^2,$$

or correspondingly we have

$$x_2 = \frac{\kappa B_1 t_1}{2[2]_q^{\eta+1}}, \quad (12)$$

and

$$x_3 = \frac{\kappa B_1}{2[3]_q^{\eta+1}}\left(t_2 - \frac{t_1^2}{2}\right) + \frac{\kappa B_2 t_1^2}{4[3]_q^{\eta+1}}. \quad (13)$$

Next, by (12) and (13), we have

$$x_3 - \nu x_2^2 = \frac{\kappa B_1}{2[3]_q^{\eta+1}}\left(t_2 - t_1^2\left(\frac{1}{2} - \frac{B_2}{2B_1} + \frac{\nu\kappa [3]_q^{\eta+1} B_1}{2[2]_q^{2\eta+2}}\right)\right). \quad (14)$$

Let

$$\zeta = \frac{1}{2}\left(1 - \frac{B_2}{B_1} + \frac{\nu\kappa [3]_q^{\eta+1} B_1}{[2]_q^{2\eta+2}}\right).$$

Therefore, (14) can be written as

$$x_3 - \nu x_2^2 = \frac{\kappa B_1}{2[3]_q^{\eta+1}}\left(t_2 - \zeta t_1^2\right).$$

By implementing Lemma 1, it appears that

$$|x_3 - \nu x_2^2| \leq \frac{|\kappa B_1|}{[3]_q^{\eta+1}} \max\left\{1; \left|\frac{B_2}{B_1} - \frac{\nu\kappa [3]_q^{\eta+1} B_1}{[2]_q^{2\eta+2}}\right|\right\}.$$

Theorem 1 has been successfully demonstrated.

By adopting $\kappa = 1$ and $\eta = 0$ into Theorem 1, we attained the following corollary.

Corollary 1 (Janteng et al., 2020) Let $\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \ldots$ with $B_1 \neq 0$, and L as interpreted by (1) belongs to the class $J^{0}_{q,1}(\varphi)$ and $\nu \in \mathbb{C}$, then

$$|x_3 - \nu x_2^2| \leq \frac{B_1}{[3]_q} \max\left\{1, \left|\frac{B_2}{B_1} - \frac{\nu B_1 [3]_q}{[2]_q^2}\right|\right\}.$$

Next, we have the next theorem by utilizing Lemma 2.

Theorem 2 Let $\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \ldots$ where $B_1 > 0$ and $B_2 \geq 0$. Let

$$\varrho_1 = \frac{(B_2 - B_1)[2]_q^{2\eta+2}}{\kappa B_1^2 [3]_q^{\eta+1}},$$

$$\varrho_2 = \frac{(B_1 + B_2)[2]_q^{2\eta+2}}{\kappa B_1^2 [3]_q^{\eta+1}},$$

and

$$\varrho_3 = \frac{B_2[2]_q^{2\eta+2}}{\kappa B_1^2 [3]_q^{\eta+1}}.$$

Let L interpreted by (1) belongs to the class $J^n_{q,b}(\varphi)$. Therefore,

$$|x_3 - \nu x_2^2| \leq \begin{cases} \frac{\kappa B_2}{[3]_q^{\eta+1}} - \frac{\nu\kappa^2 B_1^2}{[2]_q^{2\eta+2}}, & \nu \leq \varrho_1; \\ \frac{\kappa B_1}{[3]_q^{\eta+1}}, & \varrho_1 \leq \nu \leq \varrho_2; \\ \frac{-\kappa B_2}{[3]_q^{\eta+1}} + \frac{\nu\kappa^2 B_1^2}{[2]_q^{2\eta+2}}, & \nu \geq \varrho_2. \end{cases}$$

First, let $\nu \leq \varrho_1$, subsequently

$$|x_3 - \nu x_2^2| \leq \frac{\kappa B_1}{2[3]_q^{\eta+1}}(-4\zeta + 2) \leq \frac{\kappa B_2}{[3]_q^{\eta+1}} - \frac{\nu\kappa^2 B_1^2}{[2]_q^{2\eta+2}}.$$

Let $\varrho_1 \leq \nu \leq \varrho_2$ and utilizing the calculation above, we have

$$|x_3 - \nu x_2^2| \leq \frac{\kappa B_1}{[3]_q^{\eta+1}}.$$

If $\nu \geq \varrho_2$, subsequently

$$|x_3 - \nu x_2^2| \leq \frac{\kappa B_1}{2[3]_q^{\eta+1}}(4\zeta - 2) \leq -\frac{\kappa B_2}{[3]_q^{\eta+1}} + \frac{\nu\kappa^2 B_1^2}{[2]_q^{2\eta+2}}.$$

Moreover, if $\varrho_1 \leq \nu \leq \varrho_3$, subsequently

$$|x_3 - \nu x_2^2| + \frac{[2]_q^{2\eta+2}}{\kappa B_1^2 [3]_q^{\eta+1}}\left(B_1 + B_2 - \frac{\nu\kappa B_1^2 [3]_q^{\eta+1}}{[2]_q^{2\eta+2}}\right)|x_2|^2 \leq \frac{\kappa B_1}{[3]_q^{\eta+1}},$$

and if $\varrho_3 \leq \nu \leq \varrho_2$, subsequently

$$|x_3 - \nu x_2^2| + \frac{[2]_q^{2\eta+2}}{\kappa B_1^2 [3]_q^{\eta+1}}\left(B_1 - B_2 + \frac{\nu\kappa B_1^2 [3]_q^{\eta+1}}{[2]_q^{2\eta+2}}\right)|x_2|^2 \leq \frac{\kappa B_1}{[3]_q^{\eta+1}}.$$

Proof. By utilizing Lemma 2 in (14), we attain the outcomes in Theorem 2.

Moreover, we have established some theorems for the class $W^{\eta}_{(q,\kappa)}(\varphi)$.

Theorem 3 Let $\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \ldots$ with $B_1 \neq 0$, and L is interpreted by (1) belongs to the class $W^{\eta}_{q,\kappa}(\varphi)$ and $\nu \in \mathbb{C}$, then

$$|x_3 - \nu x_2^2|$$

$$\leq \frac{|\kappa B_1|}{(1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}}$$

$$\max\left\{1; \left|-\frac{\kappa B_1\left((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3} - \nu\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right)\right)}{((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2})^2} - \frac{B_2}{B_1}\right|\right\}.$$

Proof. If $L \in W^{\eta}_{q,\kappa}(\varphi)$, then a Schwarz function $y(\xi)$ exists where $y(0) = 0$ and $|y(\xi)| < 1$ in D such that

$$1 + \frac{1}{\kappa}\left((1-\delta)\frac{\xi D_q M_q^{\eta} L(\xi)}{M_q^{\eta} L(\xi)} + \delta\left(1 + \frac{q\xi D_q\left(D_q M_q^{\eta} L(\xi)\right)}{D_q\left(M_q^{\eta} L(\xi)\right)}\right) - 1\right) = \varphi(y(\xi)), \tag{15}$$

Let

$$h(\xi) = 1 + \frac{1}{\kappa}\left((1-\delta)\frac{\xi D_q M_q^{\eta} L(\xi)}{M_q^{\eta} L(\xi)} + \delta\left(1 + \frac{q\xi D_q\left(D_q M_q^{\eta} L(\xi)\right)}{D_q\left(M_q^{\eta} L(\xi)\right)}\right) - 1\right) = 1 + d_1\xi + d_2\xi^2 + \ldots \tag{16}$$

By linking (6), (15), and (16), it results in (8). Now, by linking (1), (2), and (3), we obtain that

$$h(\xi) = 1 + \frac{1}{\kappa}\left((1-\delta)([2]_q - 1)[2]_q^{\eta} x_2 + \delta q[2]_q^{\eta+2} x_2\right) \xi + \frac{1}{\kappa}\left((1-\delta)([3]_q - 1)[3]_q^{\eta} x_3 - (1-\delta)([2]_q - 1)[2]_q^{2\eta} x_2^2 + \delta q[3]_q^{\eta+2} x_3 - \delta q x_2^2 [2]_q^{2\eta+3}\right)\xi^2. \tag{17}$$

By linking (10) and (17), we get that

$$d_1 = \frac{x_2}{\kappa}\left((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2}\right) = \frac{1}{2}B_1 t_1,$$

and

$$d_2 = \frac{x_3}{\kappa}\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right) - \frac{x_2^2}{\kappa}\left((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3}\right)$$

$$= \frac{1}{2}B_1\left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}B_2 t_1^2.$$

or correspondingly we have

$$x_2 = \frac{\kappa B_1 t_1}{2\left((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2}\right)}. \tag{18}$$

$$x_3 = \frac{1}{(1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}} \left(\frac{\kappa^2 B_1^2 t_1^2 \left((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3}\right)}{4\left((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2}\right)^2} + \frac{1}{2}\kappa B_1 \left(t_2 - \frac{t_1^2}{2}\right) + \frac{1}{4}\kappa B_2 t_1^2 \right). \tag{19}$$

By (18) and (19), we have

$$x_3 - \nu x_2^2 = \frac{\kappa B_1}{2\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right)} \left(t_2 - t_1^2 \left(- \frac{\kappa B_1\left((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3} - \nu\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right)\right)}{2\left((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2}\right)^2} - \frac{B_2}{2B_1} + \frac{1}{2}\right)\right). \tag{20}$$

Let

$$\zeta = -\kappa B_1 \frac{\left((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3} - \nu\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right)\right)}{2\left((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2}\right)^2} - \frac{B_2}{2B_1} + \frac{1}{2}.$$

Therefore, (20) can be written as

$$x_3 - \nu x_2^2 = \frac{\kappa B_1}{2\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right)} \left(t_2 - \zeta t_1^2\right).$$

By implementing Lemma 1, it appears that

$$|x_3 - \nu x_2^2| \leq \frac{|\kappa B_1|}{2\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right)} \max\left\{1; \left| - \frac{B_1}{\left((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2}\right)^2} \left((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3} - \nu\left((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}\right)\right) - \frac{B_2}{B_1}\right|\right\}$$

Theorem 3 has been successfully demonstrated.

By adopting $\eta = 1$ and $\delta = 0$ into Theorem 3, we attained the following corollary.

Corollary 2 (Seoudy and Aouf, 2016) Let $\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \ldots$ with $B_1 \neq 0$, and L interpreted by (1) belongs to the class $W^1_{q,\kappa}(\varphi)$ and $\nu \in \mathbb{C}$, then

$$|x_3 - \nu x_2^2| \leq \frac{\kappa B_1}{([3]_q - 1)[3]_q} \max\left\{1, \left|\frac{B_2}{B_1} + \frac{\kappa B_1}{([2]_q - 1)}\left(1 - \nu \frac{([3]_q - 1)[3]_q}{([2]_q - 1)([2]_q)^2}\right)\right|\right\}.$$

As a final note, we come out with the last theorem by utilizing Lemma 2.

Theorem 4 Let $\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \ldots$ where $B_1 > 0$ and $B_2 \geq 0$. Let

$$\varrho_4 = \frac{((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3})B_1^2\kappa + ((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2})^2(B_2 - B_1)}{((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2})B_1^2\kappa},$$

$$\varrho_5 = \frac{((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3})B_1^2\kappa + ((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2})^2(B_2 + B_1)}{((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2})B_1^2\kappa},$$

and

$$\varrho_6 = \frac{((1-\delta)([2]_q - 1)[2]_q^{2\eta} + \delta q[2]_q^{2\eta+3})B_1^2\kappa + ((1-\delta)([2]_q - 1)[2]_q^{\eta} + \delta q[2]_q^{\eta+2})^2 B_2}{((1-\delta)([3]_q - 1)[3]_q^{\eta} + \delta q[3]_q^{\eta+2})B_1^2\kappa}.$$

Let L interpreted by (1) belongs to the class $W^n_{q,\kappa}(\varphi)$.
Let $\nu \leq \varrho_4$, then:

$$\left|x_3 - \nu x_2^2\right| \leq \frac{\kappa B_1}{2\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)}(-4\zeta + 2) \leq \frac{\kappa B_2}{(1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}} + \frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{2\eta}+\delta q[2]_q^{2\eta+3}-\nu\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\right)\kappa^2 B_1^2}{\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}.$$

Now, let $\varrho_4 \leq \nu \leq \varrho_5$; then utilizing the calculation above, we acquire:

$$\left|x_3 - \nu x_2^2\right| \leq \frac{\kappa B_1}{(1-\delta)\left([3]_q - 1\right)[3]_q^{\eta} + \delta q[3]_q^{\eta+2}}.$$

Then,

$$\left|x_3-\nu x_2^2\right| \leq \begin{cases} \frac{\kappa B_2}{(1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}} + \frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{2\eta}+\delta q[2]_q^{2\eta+3}-\nu\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\right)\kappa^2 B_1^2}{\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}, & \nu \leq \varrho_4; \\ \frac{\kappa B_1}{(1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}}, & \varrho_4 \leq \nu \leq \varrho_5; \\ -\frac{\kappa B_2}{(1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}} - \frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{2\eta}+\delta q[2]_q^{2\eta+3}-\nu\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\right)\kappa^2 B_1^2}{\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}, & \nu \geq \varrho_5. \end{cases}$$

If $\nu \geq \varrho_5$, subsequently:

$$\left|x_3-\nu x_2^2\right|$$

$$\leq \frac{\kappa B_1}{2\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)}(4\zeta-2)$$

$$\leq -\frac{\kappa B_2}{(1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}}$$

$$-\frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{2\eta}+\delta q[2]_q^{2\eta+3}-\nu\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\right)\kappa^2 B_1^2}{\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}.$$

Moreover, if $\varrho_4 \leq \nu \leq \varrho_6$, subsequently:

$$\left|x_3-\nu x_2^2\right|$$

$$+\frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}{\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)B_1^2\kappa}$$

$$\Bigg(B_1-B_2-\kappa B_1^2$$

$$\frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{2\eta}+\delta q[2]_q^{2\eta+3}-\nu\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\right)}{\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}\Bigg)|x_2|^2$$

$$\leq \frac{\kappa B_1}{(1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}}.$$

and if $\varrho_6 \leq \nu \leq \varrho_5$, subsequently:

$$\left|x_3-\nu x_2^2\right|+\frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}{\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)B_1^2\kappa}$$

$$\Bigg(B_1+B_2+\kappa B_1^2$$

$$\frac{\left((1-\delta)\left([2]_q-1\right)[2]_q^{2\eta}+\delta q[2]_q^{2\eta+3}-\nu\left((1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}\right)\right)}{\left((1-\delta)\left([2]_q-1\right)[2]_q^{\eta}+\delta q[2]_q^{\eta+2}\right)^2}\Bigg)|x_2|^2$$

$$\leq \frac{\kappa B_1}{(1-\delta)\left([3]_q-1\right)[3]_q^{\eta}+\delta q[3]_q^{\eta+2}}.$$

Proof. By utilizing Lemma 2 in (20), we attain the outcomes in Theorem 4.

Considering $\eta = 1$ and $\delta = 0$ into Theorem 4, we attained the following corollary.

Corollary 2.3 (Seoudy and Aouf, 2016) Let $\varphi(\xi) = 1 + B_1\xi + B_2\xi^2 + B_3\xi^3 + \ldots$ where $B_1 > 0$ and $B_2 \geq 0$. Let

$$\varrho_4 = \frac{([2]_q-1)[2]_q^2 B_1^2\kappa + (([2]_q-1)[2]_q)^2(B_2-B_1)}{([3]_q-1)[3]_q B_1^2\kappa},$$

$$\varrho_5 = \frac{([2]_q-1)[2]_q^2 B_1^2\kappa + (([2]_q-1)[2]_q)^2(B_2+B_1)}{([3]_q-1)[3]_q B_1^2\kappa},$$

and

$$\varrho_6 = \frac{(\kappa B_1^2+([2]_q-1)B_2)([2]_q-1)[2]_q^2}{([3]_q-1)B_1^2\kappa}.$$

Let L interpreted by (1) belongs to the class $W_{q,\kappa}^{1}(\varphi)$. Then,

$$|x_3-\nu x_2^2| \leq \begin{cases} \frac{\kappa B_2}{([3]_q-1)[3]_q} + \frac{\kappa^2 B_1^2}{([2]_q-1)([3]_q-1)[3]_q}\left(1-\frac{([3]_q-1)[3]_q}{([2]_q-1)[2]_q^2}\nu\right), & \nu \leq \varrho_4; \\ \frac{\kappa B_1}{([3]_q-1)[3]_q}, & \varrho_4 \leq \nu \leq \varrho_5; \\ -\frac{\kappa B_2}{([3]_q-1)[3]_q} - \frac{\kappa^2 B_1^2}{([2]_q-1)([3]_q-1)[3]_q}\left(1-\frac{([3]_q-1)[3]_q}{([2]_q-1)[2]_q^2}\nu\right), & \nu \geq \varrho_5. \end{cases}$$

Further, if $\varrho_4 \leq \nu \leq \varrho_6$, subsequently

$$|x_3-\nu x_2^2| + \frac{([2]_q-1)^2[2]_q^2}{([3]_q-1)[3]_q B_1^2\kappa}$$

$$\left(B_1-B_2-\frac{\kappa B_1^2}{([2]_q-1)}\left(1-\frac{([3]_q-1)[3]_q}{([2]_q-1)[2]_q^2}\nu\right)\right)|x_2|^2$$

$$\leq \frac{B_1\kappa}{([3]_q-1)[3]_q}.$$

If $\varrho_6 \leq \nu \leq \varrho_5$, subsequently

$$|x_3-\nu x_2^2| + \frac{([2]_q-1)^2[2]_q^2}{([3]_q-1)[3]_q B_1^2\kappa}$$

$$\left(B_1+B_2+\frac{\kappa B_1^2}{([2]_q-1)}\left(1-\frac{([3]_q-1)[3]_q}{([2]_q-1)[2]_q^2}\nu\right)\right)|x_2|^2$$

$$\leq \frac{\kappa B_1}{([3]_q-1)[3]_q}.$$

4. CONCLUSION

As has been demonstrated, we have established the new subclasses $J^{\eta}_{q,\kappa}(\varphi)$ and $W^{\eta}_{q,\kappa}(\varphi)$. Last but not least, we have successfully attained x_2 and x_3, and the upper bound for the functional $|x_3 - vx_2^2|$ of the functions L that are in the new subclasses.

5. ACKNOWLEDGMENT

The study of this paper was partially funded by GUG0673-1/2024.

REFERENCES

Ahmad, B., M. G. Khan, B. A. Frasin, M. K. Aouf, T. Abdeljawad, W. K. Mashwani, and M. Arif (2021). On q-Analogue Of Meromorphic Multivalent Functions In Lemniscate Of Bernoulli Domain. *AIMS Mathematics*, **6**(4); 3037–3052

Akca, H., J. Benbourenane, and H. Eleuch (2019). The q-Derivative And Differential Equation. *Journal of Physics: Conference Series*, **1411**; 012002

Al-Shbeil, I., M. I. Faisal, M. Arif, M. Abbas, and R. K. Alhefthi (2023). Investigation of the Hankel Determinant Sharp Bounds for a Specific Analytic Function Linked to a Cardioid-Shaped Domain. *Mathematics*, **11**(17); 3664

Alsoboh, A. and M. Darus (2019). On Fekete-Szegö Problem Associated With q-Derivative Operator. *Journal of Physics: Conference Series*, **1212**; 012003

Amini, E., S. Al-Omari, K. Nonlaopon, and D. Baleanu (2022). Estimates For Coefficients Of Bi-Univalent Functions Associated With A Fractional q-Difference Operator. *Symmetry*, **14**(5); 879

Andrei, L. and V. Cuas (2021). A Generalized Class Of Functions Defined By The q-Difference Operator. *Symmetry*, **13**(12); 2361

Ayinla, R. O. and T. O. Opoola (2019). The Fekete-Szegö Functional And Second Hankel Determinant For A Certain Subclass Of Analytic Functions. *Applied Mathematics*, **10**; 1071–1078

Fekete, M. and G. Szeö (1933). Eine Bermerkung Über Ungerade Schlichte Funktionen. *J. London Math. Soc.*, **8**(2); 85–89

Hern, A. L. P., A. Janteng, and R. Omar (2022). Subclasses Of Analytic Functions With Negative Coefficients Involving q-Derivative Operator. *Science and Technology Indonesia*, **7**(3); 327–332

Hu, Q., T. G. Shaba, J. Younis, B. Khan, W. K. Mashwani, and M. Caglar (2022). Applications Of q-Derivative Operator To Subclasses Of Bi-Univalent Functions Involving Gegenbauer Polynomials. *Applied Mathematics in Science and Engineering*, **30**(1); 501–520

Hussain, S., S. Khan, M. A. Zaighum, and M. Darus (2017). Certain Subclass Of Analytic Functions Related With Conic Domains And Associated With Salagean q-Differential Operator. *AIMS Mathematics*, **2**(4); 622–634

Hussain, S., S. Khan, M. A. Zaighum, and M. Darus (2018). Applications Of A q-Salagean Type Operator On Multivalent Functions. *Journal of Inequalities and Applications*, **2018**; 1–12

Jackson, F. H. (1909). On q-Functions And A Certain Difference Operator. *Transactions of the Royal Society of Edinburgh*, **46**(2); 253–281

Jackson, F. H. (1910). On q-Definite Integrals. *The Quarterly Journal of Pure and Applied Mathematics*, **41**; 193–203

Janteng, A., A. P. H. Liew, and R. Omar (2020). Fekete-Szegö Functional Of Classes Of Analytic Functions Involving The q-Derivative Operator. *Applied Mathematical Sciences*, **14**(10); 481–488

Khan, M. G., B. Ahmad, G. Murugusundaramoorthy, and S. Nadeem (2022). Third Hankel Determinant And Zalcman Functional For A Class Of Starlike Functions With Respect To Symmetric Points Related With Sine Function. *AIMS Mathematics*, **7**(4); 5405–5423

Khan, N., B. Khan, Q. Z. Ahmad, and S. Ahmad (2017). Some Convolution Properties Of Multivalent Analytic Functions. *AIMS Mathematics*, **2**(2); 260–268

Kowalczyk, B. and A. Lecko (2014). The Fekete-Szegö Inequality For Close-To-Convex Functions With Respect To A Certain Starlike Function Dependent On A Real Parameter. *Journal of Inequalities and Applications*, **2014**; 1–16

Ma, W. and M. Minda (1992). A Unified Treatment Of Some Special Classes Of Univalent Functions. In *International Conference on Complex Analysis at the Nankai Institute of Mathematics*. pages 157–169

Naeem, M., S. Hussain, T. Mahmood, S. Khan, and M. Darus (2019). A New Subclass Of Analytic Functions Defined By Using Salagean q-Differential Operator. *Mathematics*, **7**(5); 458

Oros, G. I. (2023). New Developments In Geometric Function Theory. *Axioms*, **12**(1); 59

Ramachandran, C., T. Soupramanien, and B. A. Frasin (2017). New Subclasses Of Analytic Function Associated With q-Difference Operator. *European Journal of Pure and Applied Mathematics*, **10**(2); 348–362

Saliu, A., K. I. Noor, S. Hussain, and M. Darus (2020). On Quantum Differential Subordination Related With Certain Family Of Analytic Functions. *Journal of Mathematics*, **2021**(1); 6675732

Seoudy, T. M. and M. K. Aouf (2016). Coefficient Estimates Of New Classes Of q-Starlike And q-Convex Functions Of Complex Order. *Journal of Mathematical Inequalities*, **10**(1); 135–145

Shaba, T. G., S. Araci, B. O. Adebesin, F. Tchier, S. Zainab, and B. Khan (2023). Sharp Bounds Of The Fekete-Szegö Problem And Second Hankel Determinant For Certain Bi-Univalent Functions Defined By A Novel q-Differential Operator Associated With q-Limacon Domain. *Fractal Fract.*, **7**(7); 506

Srivastava, H. M., S. Hussain, A. Raziq, and M. Raza (2018). The Fekete-Szegö Functional For A Subclass Of Analytic Functions Associated With Quasi-Subordination. *Carpathian Journal of Mathematics*, **34**(1); 103–113

Srivastava, H. M., B. Khan, N. Khan, M. Tahir, S. Ahmad, and N. Khan (2021). Upper Bound Of The Third Hankel Determinant For A Subclass Of q-Starlike Functions Associated With The q-Exponential Function. *Bulletin des Sciences Mathematiques*, **167**; 102942

Yie, T. S. and A. Janteng (2024). Fekete-Szegö Functional For Classes $X_q^n(\varphi)$ And $Y_q^n(\varphi)$. *Malaysian Journal of Fundamental and Applied Sciences*, **20**; 435–443

Zainab, S., M. Raza, Q. Xin, M. Jabeen, S. N. Malik, and S. Riaz (2021). On q-Starlike Functions Defined By q-Ruscheweyh Differential Operator In Symmetric Conic Domain. *Symmetry*, **13**(10); 1947

