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Research Paper

Zagreb-Based Indices of Line Graph of Prime Coprime Graph for Integers Modulo Group

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Abstract

In this paper, we investigate the line graph of the prime coprime graph associated with the integers modulo group. Explicit general formulas are derived for the first Zagreb index, the second Zagreb index, and the hyper-Zagreb index of the considered structures. A comparative analysis is performed between the newly obtained results and previously reported findings, highlighting structural differences and index growth behaviour under the line-graph transformation. Furthermore, a statistical analysis is conducted to explore the quantitative relationship between the prime coprime graph and its corresponding line graph with respect to the computed Zagreb-based indices. The results provide deeper insight into the structural complexity of algebraically defined graphs and clarify how degree-based topological descriptors evolve under graph transformations.

Keywords

Line Graph, Prime Coprime Graph, Integers Modulo Group, Zagreb-Based Indices

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1. INTRODUCTION

Degree-based topological indices play a central role in relating graph structure to chemical and physicochemical properties. Among the most prominent are the Zagreb family of indices, including the first Zagreb index, the second Zagreb index, and the hyper-Zagreb index. These indices quantify local degree interactions and thereby capture structural complexity and molecular connectivity. Related graph invariants such as the locating chromatic number also describe vertex position in graphs (Abel et al., 2025). Their applications span drug design, synthesis planning, and property prediction (Hassan, 2023). Related work on the codification of chemical connectivity for small-sized drugs via topological indices is reported in (Gonzalez-Diaz et al., 2007). The predictive strength of Zagreb-type descriptors is further supported by the use of general multiplicative Zagreb indices to estimate enthalpies of formation for hydrocarbons, particularly on a dataset of 25 benzenoid compounds (Noureen et al., 2024). Similarly, several degree-based descriptors, including the Zagreb and Randic indices, have been applied to analyze structural complexity and electronic characteristics of non-kokule benzenoid systems, which are known for their behavior and potential applications in molecular design (Yow et al., 2026).

Research on connection-based Zagreb descriptors includes investigations of the expected value of the first Zagreb connection index for random cyclooctatetraene chains, random polyphenyl chains, and random chain networks (Raza et al., 2023), as well as studies on polycyclic aromatic hydrocarbon structures (Arockiaraj et al., 2023; Usman and Javaid, 2024). Related theoretical work also derives formula for Zagreb index, in Cayley digraphs and shows that these indices grow linearly with the structural size of the graph (Pongpipat and Nupo, 2026). Beyond chemical systems, the hyper-Zagreb index has also been benchmarked against three classical Zagreb indices in the statistical analysis of “Murder” crime patterns in India (Islam et al., 2024), illustrating the broader methodological applicability of Zagreb-based measures in complex network analysis.

A complementary line of research investigates topological indices on transformed graph structures, particularly those arising from line-graph constructions. Eccentricity-based indices have been computed for para-line graphs of certain hexagonal cactus chains (Durgut and Turacı, 2023), while reverse degree-based indices on line graphs have been explored in applications to coronavirus drug design (Rosary, 2022). Extremal properties under the super line-graph operation have been analyzed through the generalized Randic index (Ul Ain et al., 2024). From a spectral perspective, the characterization of extremal eigenvalues of line graphs, once regarded as challenging (Chang et al., 2024), has been completely resolved (Zhen et al., 2025). Related contributions include best-possible bounds for the arithmetic-geometric index (Li and Zhang, 2022) and a comprehensive study of the general sum-connectivity index of line graphs, together with a classification of the extremal graphs (Chen, 2023). These results collectively demonstrate that line-graph transformations significantly enrich the structural and behavior of degree-based and connectivity-based indices.

Motivated by these developments, the present study focuses on Zagreb-based indices within a structured algebraic graph family, namely the prime coprime graph of the integers modulo group and its associated line graph. Studies on the orders of elements in groups provide an important algebraic foundation for graph structures derived from group operations (Mannan et al., 2022). Related algebraic studies have also extended commuting mapping from ring structures to modules using the concept of idealization (Fitriani et al., 2025). The prime coprime graph has been investigated in terms of its-Zagreb-type properties (Abdurahim et al., 2025) and spectral characteristics (Romdhini et al., 2025).

Although Zagreb-type indices for the prime coprime graph of the integers modulo group have recently been investigated (Abdurahim et al., 2026), the behavior of these indices under graph transformations has not yet been explored. In particular, the line graph transformation fundamentally alters the degree structure of the original graph by converting edges into vertices, which consequently changes the interaction patterns among vertex degrees. Therefore, the Zagreb indices of the resulting line graph cannot be directly inferred from those of the original prime coprime graph and require a separate theoretical derivation. To the best of our knowledge, no previous study has established explicit formulas for Zagreb-based indices of the line graph associated with the prime coprime graph of the integers modulo group. This gap motivates the present work. Recall that the line graph of the prime coprime graph is constructed by representing each edge of the original graph as a vertex; two vertices in the line graph are adjacent if and only if their corresponding edges in the prime coprime graph share a common endpoint. In contrast to existing studies on Zagreb indices of algebraic graphs, this work derives the explicit formulas for the first Zagreb, second Zagreb, and hyper Zagreb indices for the line graph of the prime coprime graph of the integers modulo group and analyzes the structural behavior. Studying Zagreb-based indices on both the original and line-graph structures enables a deeper understanding of how algebraic constraints influence degree-based complexity measures under graph transformation.

In addition to deriving explicit formulations and structural properties, this paper incorporates a statistical analysis to further elucidate the interrelationships among Zagreb-based indices. Such statistical perspectives contribute to the refinement of quantitative structure-property relationship (QSPR) models and support the development of statistical-mechanical descriptors of molecular graphs (Estrada, 2022). Logarithmic regression analyses conducted using SPSS have revealed strong and statistically significant correlations between proposed topological indices and entropy-based measures (Ahmed et al., 2025). Building upon these findings, multilinear regression models have demonstrated that topological indices of selected antiviral compounds exhibit strong associations with physicochemical descriptors (Manonmani and Selvarani, 2025). Furthermore, additional regression modeling has confirmed meaningful relationships between these indices and pharmacological properties (Gayathri and Roy, 2025). These results indicate that degree-based descriptors such as Zagreb indices can serve as useful variables in QSPR models that relate molecular structure to chemical and biological properties. These statistical validations reinforce the practical relevance of Zagreb-based descriptors beyond purely theoretical graph analysis.

2. EXPERIMENTAL SECTION

In this section, we fix notation and recall the basic objects and indices used throughout the paper.

Definition 2.1. *(Adhikari and S. Banerjee, 2021) Let* G *be a finite group. The prime coprime graph of* G*, denoted by* Γ_G*, is the graph whose vertex set is* G*. Two distinct vertices* $a, b \in G$ *are adjacent if and only if* $(|u|, |v|) = 1$ *or* $(|u|, |v|) = p$*, where* p *is a prime number.*

We now recall the Zagreb-based indices for the graph $G = (V, E)$. Let $\deg(v)$ denote the degree of v in V. The first Zagreb index is the sum of squared vertex degrees (Gutman and Das, 2004):

$$M_1(G) = \sum_{v \in V} (\deg(v))^2 \tag{1}$$

The second Zagreb index is the sum, over all edges (u, v), of the product of the endpoint degrees (Das et al., 2015):

$$M_2(G) = \sum_{(uv,) \in E} \deg(u) \cdot \deg(v) \tag{2}$$

The hyper-Zagreb index is the sum, over all edges (u, v), of the square of the sum of the endpoint degrees (Shirdel et al., 2013):

$$HM(G) = \sum_{(u,v) \in E} (\deg(u) + \deg(v))^2 \tag{3}$$

Throughout this paper, let $\Gamma_{\mathbb{Z}_n}$ denote the prime coprime graph on the integers modulo n. The vertex set and edge set of $\Gamma_{\mathbb{Z}_n}$ are denoted by $V(\Gamma_{\mathbb{Z}_n})$ and $E(\Gamma_{\mathbb{Z}_n})$, respectively. The following results was established in (Abdurahim et al., 2025).

Theorem 2.2. *(Abdurahim et al., 2025) Let* $n = p^k$*, where* p *is a prime number and* $k \geq 2$ *is an integer. Then the number of edges of* $\Gamma_{\mathbb{Z}_n}$ *is given by*

$$\left|E\left(\Gamma_{\mathbb{Z}_n}\right)\right| = \frac{1}{2}\left(2p^{k+1} - p^2 - p\right).$$

We next summarize prior results on Zagreb-based indices for $\Gamma_{\mathbb{Z}_n}$, for $n = p^k$ with p prime and integer $k \geq 2$, which will be used in the subsequent sections.

Theorem 2.3. *(Abdurahim et al., 2025) The first Zagreb index of $\Gamma_{\mathbb{Z}_n}$ is*

$$M_1\left(\Gamma_{\mathbb{Z}_n}\right) = p\left(p^{2k} + p^{k+1} - 2p^k - p^2 + 1\right).$$

Theorem 2.4. *(Abdurahim et al., 2025) The second Zagreb index of $\Gamma_{\mathbb{Z}_n}$ is*

$$M_2\left(\Gamma_{\mathbb{Z}_n}\right) = \frac{1}{2}p(p^k - 1)\left(3p^{k+1} - p^k - 2p^2 - p + 1\right).$$

Theorem 2.5. *(Abdurahim et al., 2025) The hyper-Zagreb index of $\Gamma_{\mathbb{Z}_n}$ is*

$$\begin{aligned} HM\left(\Gamma_{\mathbb{Z}_n}\right) = {} & p^{3k+1} + 3p^{2k+2} - 4p^{2k+1} - p^{k+3} \\ & - 4p^{k+2} + 5p^{k+1} - p^4 + 2p^3 + p^2 - 2p. \end{aligned}$$

Moreover, we denote by $L(\Gamma_{\mathbb{Z}_n})$ the line graph of $\Gamma_{\mathbb{Z}_n}$. Its vertex set and edge set are denoted by $V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ and $E\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ for its vertex and edge sets, respectively. By definition, each vertex of $L(\Gamma_{\mathbb{Z}_n})$ corresponds to an edge of the prime coprime graph $\Gamma_{\mathbb{Z}_n}$. Hence, the set of vertices can be expressed as

$$V\left(L(\Gamma_{\mathbb{Z}_n})\right) = \left\{(u, v) \,\middle|\, u, v \in V\left(\Gamma_{\mathbb{Z}_n}\right),\ u \sim v\right\}.$$

Furthermore, two vertices in the line graph are adjacent if and only if the corresponding edges in $\Gamma_{\mathbb{Z}_n}$ share a common vertex. Formally, the set of edges is given by

$$E\left(L(\Gamma_{\mathbb{Z}_n})\right) = \left\{((u, v), (w, x)) \,\middle|\, \begin{array}{r} u, v, w, x \in V\left(\Gamma_{\mathbb{Z}_n}\right), \\ \{u, v\} \cap \{w, x\} \neq \emptyset \end{array}\right\}.$$

Therefore, the degree of a vertex (u, v) in the line graph $L(\Gamma_{\mathbb{Z}_n})$ is equal to the number of edges in $\Gamma_{\mathbb{Z}_n}$ that intersect the edge (u, v), that is, all edges incident to either vertex u or v, except the edge (u, v) itself. Mathematically, this can be expressed as:

$$\deg_L(u, v) = \deg(u) + \deg(v) - 2.$$

3. RESULTS AND DISCUSSIONS

Definition 3.1. *Let $\Gamma_{\mathbb{Z}_n}$ be the prime coprime graph of $\mathbb{Z}_n$. The line graph of $\Gamma_{\mathbb{Z}_n}$, denoted by $L(\Gamma_{\mathbb{Z}_n})$, is the graph whose vertex set $V\left(L(\Gamma_{\mathbb{Z}_n})\right)$ and edge set $E\left(L(\Gamma_{\mathbb{Z}_n})\right)$ are define as follows.*

$$V\left(L(\Gamma_{\mathbb{Z}_n})\right) = \left\{(u, v) \,\middle|\, u, v \in V\left(\Gamma_{\mathbb{Z}_n}\right),\ u \sim v\right\}.$$

$$E\left(L(\Gamma_{\mathbb{Z}_n})\right) = \left\{((u, v), (w, x)) \,\middle|\, \begin{array}{r} u, v, w, x \in V\left(\Gamma_{\mathbb{Z}_n}\right), \\ \{u, v\} \cap \{w, x\} \neq \emptyset \end{array}\right\}.$$

In other words, Definition 3.1 states that the vertex set of the graph $L(\Gamma_{\mathbb{Z}_n})$ corresponds to the edge set of the graph $\Gamma_{\mathbb{Z}_n}$. Moreover, two vertices in $L(\Gamma_{\mathbb{Z}_n})$ are adjacent if and only if the corresponding edges in $\Gamma_{\mathbb{Z}_n}$ share a common endpoint.

Suppose the group $\mathbb{Z}_9$ is given. The graph $\Gamma_{\mathbb{Z}_9}$ can be seen in Figure 1 of (Abdurahim et al., 2025). From the figure, the edges formed are $(0, 1)$, $(0, 2)$, $(0, 3)$, $(0, 4)$, $(0, 5)$, $(0, 6)$, $(0, 7)$, $(0, 8)$, $(1, 3)$, $(1, 6)$, $(2, 3)$, $(2, 6)$, $(3, 4)$, $(3, 5)$, $(3, 6)$, $(3, 7)$, $(3, 8)$, $(4, 6)$, $(5, 6)$, $(6, 7)$, and $(6, 8)$. These edges then become the vertices of the graph $L\left(\Gamma_{\mathbb{Z}_9}\right)$. The graph $L\left(\Gamma_{\mathbb{Z}_9}\right)$ is illustrated in Figure 1.

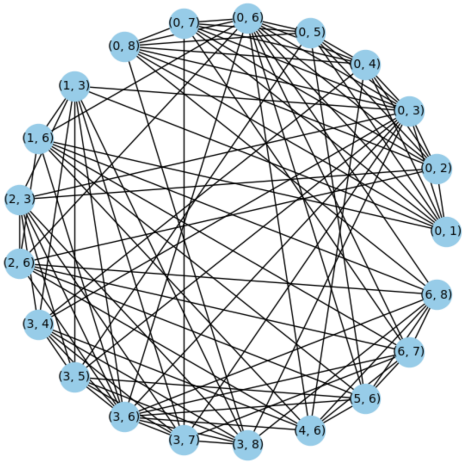

Figure 1. Line Graph for $\mathbb{Z}_9$

In this section, we present the main results concerning the Zagreb-based topological indices of the line graph of the prime coprime graph associated with the integers modulo group. We begin by determining the degree of each vertex and the total number of edges in $L(\Gamma_{\mathbb{Z}_n})$, since these structural parameters are fundamental for the computation of the first Zagreb index, the second Zagreb index, and hyper Zagreb index.

Theorem 3.2. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the line graph of the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the degree of every vertex in $L\left(\Gamma_{\mathbb{Z}_n}\right)$ is*

1. $\deg_L\left((u_1, v_1)\right) = 2(p^k - 2)$ *for* $u_1, v_1 \in V_1$ *and* $u_1 \neq v_1$.
2. $\deg_L\left((v_1, v_2)\right) = p^k + p - 3$ *for* $v_1, \in V_1$ *and* $v_2 \in V_2$.

with $V_1 = \{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\}$ *and* $V_2 = \{0, 1, 2, 3, \ldots, p^k - 1\} \backslash V_1$.

Proof. We begin by considering the vertex partition of the prime coprime graph $\Gamma_{\mathbb{Z}_n}$ where $n = p^k$. The partitions are given by $V_1 = \{0, p^{k-1}, 2p^{k-1}, 3p^{k-1}, \ldots, (p-1)p^{k-1}\}$ and $V_2 = \{0, 1, 2, 3, \ldots, p^k - 1\} \backslash V_1$. According to the definition of the prime coprime graph of the group $\mathbb{Z}_n$, the adjacency relations are as follows:

- Any two distinct vertices u_1 and v_1 are adjacent for $u_1, v_1 \in V_1$;
- A vertex $v_1 \in V_1$ is adjacent to every vertex $v_2 \in V_2$;
- Any two vertices $u_2, v_2 \in V_2$ are non-adjacent.

Observe that,

1. Because every pair of distinct vertices in V_1 is adjacent, it follows from the definition of the line graph that the vertex $(u_1, v_1) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ is adjacent to every vertex corresponding to an edge that has a non-empty intersection with $\{u_1, v_1\}$. From Theorem 2.2 in (Abdurahim

et al., 2025), we obtain that $\deg(u_1) = \deg(v_1) = p^k - 1$. This implies that both u_1 and $v - 1$ are incident with exactly $p^k - 1$ edges. Furthermore, because the vertex $\{u_1, v_1\} \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ must be distinct, it follows that u_1 and $v - 1$ are incident with exactly $p^k - 2$ edges. Hence, it follows that

$$\deg_L\left((u_1, v_1)\right) = (p^k - 2) + (p^k - 2) = 2(p^k - 2).$$

2. In this proof, two cases are considered. The first involves determining the number of edges incident to v_1 or (v_1, v_i) for each $v_i \in V$. The second is to determine the number of edges incident to v_2 or to (v_j, v_2) for each $v_j \in V_1 \backslash \{v_1\}$. As $v_1 \in V_1$ is adjacent to every vertex, it follows that v_1 is adjacent to $p^k - 1$ vertices. Thus, the edge (v_1, v_i) in $\Gamma_{\mathbb{Z}_n}$ is adjacent to $(p^k - 1) - 1 = p^k - 2$ edges. For each $v_j \in V_1 \backslash \{v_1\}$, the vertex v_2 on the edge (v_j, v_2) is adjacent to $p - 1$ edges. Hence

$$\deg_L\left((v_1, v_2)\right) = (p^k - 2) + (p - 1) = p^k + p - 3.$$

Furthermore, by the definition of a line graph, it is clear that the number of vertices of the line graph equals the number of edges in the coprime-prime graph, namely $\left|V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right| = \frac{1}{2}\left(2p^{k+1} - p^2 - p\right)$ (Abdurahim et al., 2025).

The vertex degrees obtained in Theorem 3.2 reflect the structure of the prime coprime graph for $n = p^k$. Vertices of the graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ that originate from edges of $\Gamma_{\mathbb{Z}_n}$ connecting two vertices in the set V_1 have relatively large degrees in the line graph. This occurs because each vertex in V_1 is adjacent to every vertex in the graph $\Gamma_{\mathbb{Z}_n}$. Consequently, edges involving these vertices tend to intersect with a larger number of other edges in $\Gamma_{\mathbb{Z}_n}$.

Conversely, vertices in the graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ that originate from edges connecting vertices in V_1 and V_2 have smaller degrees. This is because vertices in V_2 are adjacent only to vertices in V_1 in the graph $\Gamma_{\mathbb{Z}_n}$. Consequently, such edges share endpoints with fewer other edges in $\Gamma_{\mathbb{Z}_n}$.

The difference indicates that the line graph transformation amplifies the role of vertices with large degrees in the graph $\Gamma_{\mathbb{Z}_n}$. In other words, the connectivity structure of $\Gamma_{\mathbb{Z}_n}$ strongly influences the degree distribution in the graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$.

Theorem 3.3. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the line graph of prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the number of edges is*

$$\left|E\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right| = \frac{1}{2}\left(p^{2k+1} + p^{k+2} - 4p^{k+1} - p^3 + p^2 + 2p\right).$$

Proof. Let $u_1, v_1 \in V_1$ and $u_2, v_2 \in V_2$. By Theorem 3.2, we have $\deg_L\left((u_1, v_1)\right) = 2(p^k - 2)$ and $\deg_L\left((v_1, v_2)\right) = p^k + p - 3$. Since $|V_1| = p$, the number of vertices $(u_1, v_1) \in V_L$ with $u_1 \neq v_1$ is $C(p, 2) = \frac{1}{2}(p - 1)p$. Next, the number of vertices of the form (v_1, v_2) equals the number of edges in the prime coprime graph $\Gamma_{\mathbb{Z}_n}$ minus the number of vertices $(u_1, v_1) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$, namely

$$\frac{1}{2}\left(2p^{k+1} - p^2 - p\right) - \frac{1}{2}(p - 1)p = p^{k+1} - p^2.$$

Note that

$$\begin{aligned}
\left|E\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right| &= \frac{1}{2}\sum_{(u,v)\in V} \deg_L\left((u, v)\right) \\
&= \frac{1}{2}\left(\sum_{\substack{(u_1,v_1)\in V \\ u_1,v_1\in V_1}} \deg_L\left((u_1, v_1)\right) + \sum_{\substack{(v_1,v_2)\in V \\ v_1\in V_1, v_2\in V_2}} \deg_L\left((v_1, v_2)\right)\right) \\
&= \frac{1}{2}\left(\frac{1}{2}(p - 1)p \cdot 2\left(p^k - 2\right) + \left(p^{k+1} - p^2\right) \cdot \left(p^k + p - 3\right)\right) \\
&= \frac{1}{2}\left((p - 1)p\left(p^k - 2\right) + \left(p^{k+1} - p^2\right)\left(p^k + p - 3\right)\right) \\
&= \frac{1}{2}\left(\left(p^{k+2} - p^{k+1} - 2p^2 + 2p\right) + \left(p^{2k+1} - 3p^{k+1} - p^3 + 3p^2\right)\right) \\
&= \frac{1}{2}\left(p^{2k+1} + p^{k+2} - 4p^{k+1} - p^3 + p^2 + 2p\right),
\end{aligned}$$

and we complete the proof.

We now present a theorem that determines the first Zagreb index of the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$.

Theorem 3.4. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the first Zagreb index of $L\left(\Gamma_{\mathbb{Z}_n}\right)$ is*

$$\begin{aligned}
M_1\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) = p^{3k+1} &+ 3p^{2k+2} - 8p^{2k+1} \\
&- p^{k+3} - 8p^{k+2} + 17p^{k+1} \\
&- p^4 + 6p^3 - p^2 - 8p.
\end{aligned}$$

Proof. An argument identical to that of Theorem 3.3 yields

$$\begin{aligned}
M_1\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &= \sum_{\substack{(u_1,v_1)\in V_L \\ u_1,v_1\in V_1}} \left(\deg_L((u_1, v_1))\right)^2 + \sum_{\substack{(v_1,v_2)\in V_L \\ v_1\in V_1, v_2\in V_2}} \left(\deg_L((v_1, v_2))\right)^2 \\
&= \frac{1}{2}(p - 1)p \cdot \left(2\left(p^k - 2\right)\right)^2 + \left(p^{k+1} - p^2\right) \cdot \left(p^k + p - 3\right)^2 \\
&= \left(2p^{2k+2} - 2p^{2k+1} - 8p^{k+2} + 8p^{k+1} + 8p^2 - 8p\right) + (p^{3k+1} + \\
&\quad + p^{2k+2} - 6p^{2k+1} - p^{k+3} + 9p^{k+1} - p^4 + 6p^3 - 9p^2)
\end{aligned}$$

$$\begin{aligned}
M_1\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &= p^{3k+1} + 3p^{2k+2} - 8p^{2k+1} - p^{k+3} - 8p^{k+2} + 17p^{k+1} \\
&\quad - p^4 + 6p^3 - p^2 - 8p.
\end{aligned}$$

The formula obtained in Theorem 3.4 shows that the dominant term of $M_1\left(L(\Gamma_{\mathbb{Z}_n})\right)$ is p^{3k+1}. This indicates that the value of the first Zagreb index for the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ increases significantly compared with that of the graph $\Gamma_{\mathbb{Z}_n}$. This occurs because in the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ each vertex represents an edge of the graph $\Gamma_{\mathbb{Z}_n}$. Consequently, the degree of a vertex in the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ depends on the degrees of the two adjacent vertices in the graph $\Gamma_{\mathbb{Z}_n}$. Since many edges in $\Gamma_{\mathbb{Z}_n}$

share common endpoints, the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ exhibits a larger degree distribution.

Theorem 3.5 systematically examines the relationship between the first Zagreb index of the prime coprime graph and that of its line graph, providing a deeper understanding of how the structures of the two graphs are related.

Theorem 3.5. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, the relation between the first Zagreb index of the prime coprime graph and that of its line graph is given by*

$$M_1\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) \leq \left(p^k + 2p - 6\right) M_1\left(\Gamma_{\mathbb{Z}_n}\right).$$

Proof. By Theorem 2.3, the first Zagreb index of the prime coprime graph is $M_1\left(\Gamma_{\mathbb{Z}_n}\right) = p^{2k+1} + p^{k+2} - 2p^{k+1} - p^3 + p$. Therefore

$$\begin{aligned} p^k M_1\left(\Gamma_{\mathbb{Z}_n}\right) &= p^{3k+1} + p^{2k+2} - 2p^{2k+1} - p^{k+3} + p^{k+1} \\ pM_1\left(\Gamma_{\mathbb{Z}_n}\right) &= p^{2k+2} + p^{k+3} - 2p^{k+2} - p^4 + p^2 \end{aligned}$$

follows. Note that

$$\begin{aligned} M_1\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &= p^{3k+1} + 3p^{2k+2} - 8p^{2k+1} - p^{k+3} - 8p^{k+2} \\ &\quad + 17p^{k+1} - p^4 + 6p^3 - p^2 - 8p \\ &= \left(p^{3k+1} + p^{2k+2} - 2p^{2k+1} - p^{k+3} + p^{k+1}\right) \\ &\quad + 2\left(p^{2k+2} + p^{k+3} - 2p^{k+2} - p^4 + p^2\right) \\ &\quad - 6\left(p^{2k+1} + p^{k+2} - 2p^{k+1} - p^3 + p\right) \\ &\quad - 2p^{k+3} + 2p^{k+2} + 4p^{k+1} + p^4 - 3p^2 - 2p \\ &= p^k M_1\left(\Gamma_{\mathbb{Z}_n}\right) + 2pM_1\left(\Gamma_{\mathbb{Z}_n}\right) - 6M_1\left(\Gamma_{\mathbb{Z}_n}\right) \\ &\quad - 2p^{k+3} + 2p^{k+2}4p^{+}k + 1 + p^4 - 3p^2 - 2p \\ M_1\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &= \left(p^k + 2p - 6\right) M_1\left(\Gamma_{\mathbb{Z}_n}\right) + \left(-2p^k + p + 1\right) \\ &\quad p(p-2)(p+1). \end{aligned}$$

Clearly, $p(p-2)(p+1) \geq 0$ for prime p. We next show that $p^2 > p + 1$, equivalently $f(p) = p^2 - p - 1 > 0$. Since the quadratic has discriminant $D > 0$ and leading coefficient $a > 0$, its graph opens upward. Furthermore, one of its roots is $p_1 = \dfrac{1+\sqrt{5}}{2}$ with $p_1 > p_2$. Hence, the quadratic $f(p)$ is positive if and only if $p > \dfrac{1+\sqrt{5}}{2} \approx 1,618$. Since p is prime, we have already shown $p^2 > p + 1$. Hence, for $k \geq 2$, $2p^k \geq 2p^2 > p + 1$, so $-2p^k + p + 1 < 0$. Moreover, $p(p-2)(p+1) \geq 0$ for all primes p, with strict inequality > 0 when $p \geq 3$. Therefore, $\left(-2p^k + p + 1\right) p(p-2)(p+1) < 0$ for $p \geq 3$, and it equals to 0 when $p = 2$. Hence,

$$M_1\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) \leq \left(p^k + 2p - 6\right) M_1\left(\Gamma_{\mathbb{Z}_n}\right),$$

which completes the proof.

The following theorem provides an explicit formula for the second Zagreb index of $L\left(\Gamma_{\mathbb{Z}_n}\right)$.

Theorem 3.6. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the second Zagreb index of $L\left(\Gamma_{\mathbb{Z}_n}\right)$ is*

$$\begin{aligned} M_2\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &= \frac{1}{2}p^{4k+1} + \frac{5}{2}p^{3k+2} - 6p^{3k+1} + \frac{1}{2}p^{2k+4} - \frac{3}{2}p^{2k+3} \\ &\quad - \frac{23}{2}p^{2k+2} + \frac{43}{2}p^{2k+1} - \frac{9}{2}p^{k+4} + 16p^{k+3} + \frac{7}{2}p^{k+2} \\ &\quad - 25p^{k+1} + 7p^4 - 26p^3 + 19p^2 + 4p. \end{aligned}$$

Proof. From the definition of the prime coprime graph $\Gamma(\mathbb{Z}_n)$, every edge of $\Gamma(\mathbb{Z}_n)$ contains a vertex from V_1. Hence, every vertex of $V\left(L\left(\Gamma\left(\mathbb{Z}_n\right)\right)\right)$ involves a vertex from V_1. Therefore, checking adjacency between arbitrary vertices in the line graph $L\left(\Gamma\left(\mathbb{Z}_n\right)\right)$ reduces to considering edges incident with vertices of V_1. If $v_1 \in V_1$, then the edges of $\Gamma(\mathbb{Z}_n)$ incident with v_1 are of two types: edges (u_1, v_1) with $u_1, v_1 \in V_1$, and edges (v_1, v_2) with $v_1 \in V_1$ and $v_2 \in V_2$.

1. **Case 1.** Edges of the form u_1v_1 with $u_1, v_1 \in V_1$
   The edge u_1v_1 in $E\left(\Gamma\left(\mathbb{Z}_n\right)\right)$ corresponds to the vertex (u_1, v_1) in $V\left(L\left(\Gamma\left(\mathbb{Z}_n\right)\right)\right)$. Hence, in $L\left(\Gamma\left(\mathbb{Z}_n\right)\right)$, the vertex (u_1, v_1) is adjacent to every vertex (r_1, s_1) with $r_1, s_1 \in V_1$, $r_1 = u_1$ or $s_1 = v_1$. It is also adjacent to vertices of the form (u_1, v_2) or (u_2, v_1) with $u_2, v_2 \in V_2$.
   (a) **Case 1.1** A vertex of type $(u_1, v_1) \in Vt\left(L\left(\Gamma\left(\mathbb{Z}_n\right)\right)\right)$ is adjacent to a vertex of type $(r_1, s_1) \in V(L(\Gamma(\mathbb{Z}_n)))$
   A vertex of type (u_1, v_1) is adjacent to a vertex of type (r_1, s_1) if and only if $u_1 = r_1$ or $v_1 = s_1$; equivalently, (u_1, v_1) and (r_1, s_1) share an endpoint. Hence each (u_1, v_1) is adjacent to all (r_1, s_1)that share one endpoint with it. The number of vertices of type (u_1, v_1) is

$$C(p, 2) = \frac{1}{2}(p-1)p = \frac{1}{2}(p^2 - p).$$

   Therefore, the number of edges connecting vertices of type (u_1, v_1) to vertices of type (r_1, s_1) is

$$\begin{aligned} C\left(\frac{1}{2}(p^2 - p), 2\right) &= \frac{1}{2}\left(\frac{1}{2}(p^2 - p)\right)\left(\frac{1}{2}(p^2 - p) - 1\right) \\ &= \frac{1}{8}(p^2 - p)(p^2 - p - 2) \\ &= \frac{1}{8}(p+1)p(p-1)(p-2). \end{aligned}$$

   (b) **Case 1.2** The vertex (u_1, v_1) is adjacent to (u_1, v_2) or (u_2, v_1) for any $u_2, v_2 \in V_2$
   The number of edges connecting a vertex of type (u_1, v_1) to all vertices of type (u_1, v_2) equals the number of vertices of type (u_1, v_2), i.e., the number of edges from u_1 to all members of V_2, which is $|V_2| = p^k - p$. By the same reasoning, the number of edges connecting (u_1, v_1) to all vertices of type (u_2, v_1) is also $p^k - p$.

The number of vertices of type (u_1, v_1) is $\frac{1}{2}(p^2-p)$. Hence, the number of edges connecting a (u_1, v_1)-type vertex to vertices of type (u_1, v_2) or (u_2, v_1) is

$$\left(\frac{1}{2}(p-1)p\right)\left(\left(p^k-p\right)+\left(p^k-p\right)\right)=\left(p^k-p\right)(p-1)p.$$

2. **Case 2.** The vertex $(v_1, v_2) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ with $v_1 \in V_1$ and $v_2 \in V_2$
   A vertex of type $(v_1, v_2) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ is adjacent to every vertex of type (v_1, u_2) or (u_1, v_2) di $V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ where $u_1 \neq v_1 \in V_1$ and $u_2 \neq v_2 \in V_2$. Consequently, the total number of edges connecting vertices of type (v_1, v_2) to vertices of types (v_1, u_2) and (u_1, v_2) equals the number of such neighbors per (v_1, v_2) times the number of vertices of type (v_1, v_2).
   (a) **Case 2.1** A vertex of type $(v_1, v_2) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ is adjacent to a vertex of type (v_1, u_2).
   A vertex of type (v_1, v_2) is adjacent to every vertex of type (v_1, u_2), since they share the common endpoint $v_1 \in V_1$. The number of neighbors of this kind equals the number of edges from v_1 to V_2 excluding v_2 itself, namely

   $$|V_2|-1=p^k-p-1.$$

   Next, the number of vertices of type $(v_1, v_2) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ equals the total number of line graph vertices minus those of type (u_1, v_1) (edges inside V_1).

   $$\left|V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right|-C(p,2)=\left(\frac{1}{2}\left(2p^{k+1}-p^2-p\right)-\frac{1}{2}(p-1)p\right)=p^{k+1}-p^2.$$

   Since each vertex (v_1, v_2) is counted twice (once for each orientation), the number of edges connecting all vertices of type $(v_1, v_2) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ to vertices of type (v_1, u_2) is

   $$\frac{1}{2}\left(p^k-p-1\right)\left(p^{k+1}-p^2\right).$$

   (b) **Case 2.2** A vertex of type $(v_1, v_2) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ is adjacent to a vertex of type (u_1, v_2).
   A vertex of type (v_1, v_2) is adjacent to every vertex of type (u_1, v_2), since they share the common endpoint $v_2 \in V_2$. The number of neighbors of this kind equals the number of edges from v_2 to V_1 excluding itself, namely

   $$|V_1|-1=p-1.$$

   Since each (v_1, v_2) is counted twice (once for each orientation), the number of edges connecting all vertices of type $(v_1, v_2) \in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$ to vertices of type (v_1, u_2) is

   $$\frac{1}{2}(p-1)\left(p^{k+1}-p^2\right).$$

   Therefore, the total number of edges incident to all vertices of type (v_1, v_2) is

$$\frac{1}{2}\left(p^k-p-1\right)\left(p^{k+1}-p^2\right)+\frac{1}{2}(p-1)\left(p^{k+1}-p^2\right)=\frac{1}{2}\left(p^k-2\right)\left(p^{k+1}-p^2\right).$$

Observe that,

$$M_2\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)=\sum_{v\in V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)}\sum_{\{e,f\}\subseteq E_v}\deg_L(e)\deg_L(f),$$

where E_v is the set of edges of $\Gamma_{\mathbb{Z}_n}$ incident with v. Hence,

$$\begin{aligned}
M_2\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)&=\left(\frac{1}{8}(p+1)p(p-1)(p-2)\right)\deg_L((u_1,v_1))\deg_L((r_1,s_1))\\
&\quad+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)\deg_L((u_1,v_1))\deg_L((u_1,v_2))\\
&\quad+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)\deg_L((u_1,v_1))\deg_L((u_2,v_1))\\
&\quad+\left(\frac{1}{2}\left(p^k-p-1\right)\left(p^{k+1}-p^2\right)\right)\deg_L((v_1,v_2))\deg_L((v_1,u_2))\\
&\quad+\left(\frac{1}{2}(p-1)\left(p^{k+1}-p^2\right)\right)\deg_L((v_1,v_2))\deg_L((u_1,v_2))\\
&=\left(\frac{1}{8}(p+1)p(p-1)(p-2)\right)\cdot 2\left(p^k-2\right)\cdot 2\left(p^k-2\right)\\
&\quad+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)\cdot 2\left(p^k-2\right)\left(p^k+p-3\right)\\
&\quad+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)\cdot 2\left(p^k-2\right)\left(p^k+p-3\right)\\
&\quad+\left(\frac{1}{2}\left(p^k-p-1\right)\left(p^{k+1}-p^2\right)\right)\cdot\left(p^k+p-3\right)\left(p^k+p-3\right)\\
&\quad+\left(\frac{1}{2}(p-1)\left(p^{k+1}-p^2\right)\right)\cdot\left(p^k+p-3\right)\left(p^k+p-3\right)\\
&=\left(\frac{1}{2}(p+1)p(p-1)(p-2)\right)\left(p^k-2\right)^2+\left(\left(p^k-p\right)(p-1)p\right)\\
&\quad\cdot 2\left(p^k-2\right)\left(p^k+p-3\right)+\frac{1}{2}\left(p^k-2\right)\left(p^{k+1}-p^2\right)\cdot\left(p^k+p-3\right)^2\\
&=\left(\frac{1}{2}p^{2k+4}-p^{2k+3}-\frac{1}{2}p^{2k+2}+p^{2k+1}-2p^{k+4}+4p^{k+3}+2p^{k+2}\right.\\
&\quad\left.-4p^{k+1}+2p^4-4p^3-2p^2+4p\right)+\left(2p^{3k+2}-2p^{3k+1}-10p^{2k+2}\right.\\
&\quad\left.+10p^{2k+1}-2p^{k+4}+8p^{k+3}+6p^{k+2}-12p^{k+1}+4p^4-16p^3+12p^2\right)\\
&\quad+\left(\frac{1}{2}p^{4k+1}+\frac{1}{2}p^{3k+2}-4p^{3k+1}-\frac{1}{2}p^{2k+3}-p^{2k+2}+\frac{21}{2}p^{2k+1}\right.\\
&\quad\left.-\frac{1}{2}p^{k+4}+4p^{k+3}-\frac{9}{2}p^{k+2}-9p^{k+1}+p^4-6p^3+9p^2\right)
\end{aligned}$$

$$\begin{aligned}
M_2\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)&=\frac{1}{2}p^{4k+1}+\frac{5}{2}p^{3k+2}-6p^{3k+1}+\frac{1}{2}p^{2k+4}-\frac{3}{2}p^{2k+3}-\frac{23}{2}p^{2k+2}+\\
&\quad\frac{43}{2}p^{2k+1}-\frac{9}{2}p^{k+4}+16p^{k+3}+\frac{7}{2}p^{k+2}-25p^{k+1}+7p^4-26p^3+\\
&\quad19p^2+4p.
\end{aligned}$$

The result in Theorem 3.6 shows that the second Zagreb index of the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ is influenced by the interaction pattern among edges in the prime coprime graph $\Gamma_{\mathbb{Z}_n}$. In contrast to the first Zagreb index, which depends only on vertex degrees, the second Zagreb index takes into account the product of the degrees of adjacent vertex pairs.

In the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$, each vertex represents an edge of the graph $\Gamma_{\mathbb{Z}_n}$. Therefore, adjacency in the graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ corresponds to pairs of edges in $\Gamma_{\mathbb{Z}_n}$ that share a common endpoint. Since many edges in the graph $\Gamma_{\mathbb{Z}_n}$ are incident to vertices with large degrees, the corresponding pairs of vertices in the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$ also tend to have large degrees.

In contrast to the previous study (Abdurahim et al., 2025), which directly investigated the prime coprime graph $\Gamma_{\mathbb{Z}_n}$ with $V\left(\Gamma_{\mathbb{Z}_n}\right) = \mathbb{Z}_n$ and $E\left(\Gamma_{\mathbb{Z}_n}\right) = \{uv \mid \gcd(u,v) = 1 \text{ or prime}\}$, the present study considers the line graph $L(\Gamma_{\mathbb{Z}_n})$, where $V(L(\Gamma_{\mathbb{Z}_n})) = E(\Gamma_{\mathbb{Z}_n})$ and

$$E\left(L(\Gamma_{\mathbb{Z}_n})\right) = \left\{\left((u,v),(w,x)\right) \middle| u,v,w,x \in V\left(\Gamma_{\mathbb{Z}_n}\right),\ \{u,v\} \cap \{w,x\} \neq \emptyset\right\}.$$

This transformation increases both the number of vertices and the density of the graph, since each edge of $\Gamma_{\mathbb{Z}_n}$ becomes a vertex in $L(\Gamma_{\mathbb{Z}_n})$, and adjacency in the line graph represents the interactions between edges of the original graph. Furthermore, two vertices in the line graph are adjacent if and only if the corresponding edges in $\Gamma_{\mathbb{Z}_n}$ share a common vertex.

Theorem 3.7. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the prime coprime line graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, the relation between the second Zagreb index of the prime coprime graph and that of its line graph is given by*

$$M_2\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) > \left(\frac{5}{3}p^k + \frac{1}{3}p^2 + \frac{2}{9} - \frac{145}{27}\right) M_2\left(\Gamma_{\mathbb{Z}_n}\right) - \frac{65}{18}p^{k+4} - \frac{1015}{54}p^{k+1} - \frac{1}{3}p^5$$

Proof. By Theorem 2.4, the second Zagreb index of the prime coprime graph is $M_2\left(\Gamma_{\mathbb{Z}_n}\right) = \frac{3}{2}p^{2k+2} - \frac{1}{2}p^{2k+1} - p^{k+3} - 2p^{k+2} + p^{k+1} + p^3 + \frac{1}{2}p^2 - \frac{1}{2}p$. It follows that

$$\begin{aligned}
-\frac{145}{27}M_2\left(\Gamma_{\mathbb{Z}_n}\right) &= -\frac{145}{18}p^{2k+2} + \frac{145}{54}p^{2k+1} + \frac{145}{27}p^{k+3} + \frac{290}{27}p^{k+2} - \frac{145}{27}p^{k+1} - \\
&\quad \frac{145}{27}p^3 - \frac{145}{54}p^2 + \frac{145}{54}p \\
\frac{2}{9}pM_2\left(\Gamma_{\mathbb{Z}_n}\right) &= \frac{1}{3}p^{2k+3} - \frac{1}{9}p^{2k+2} - \frac{2}{9}p^{k+4} - \frac{4}{9}p^{k+3} + \frac{2}{9}p^{k+2} + \frac{2}{9}p^4 + \frac{1}{9}p^3 - \frac{1}{9}p^2 \\
\frac{1}{3}p^2M_2\left(\Gamma_{\mathbb{Z}_n}\right) &= \frac{1}{2}p^{2k+4} - \frac{1}{6}p^{2k+3} - \frac{1}{3}p^{k+5} - \frac{2}{3}p^{k+4} + \frac{1}{3}p^{k+3} + \frac{1}{3}p^5 + \frac{1}{6}p^4 - \frac{1}{6}p^3 \\
\frac{5}{3}p^kM_2\left(\Gamma_{\mathbb{Z}_n}\right) &= \frac{5}{2}p^{3k+2} - \frac{5}{6}p^{3k+1} - \frac{5}{3}p^{2k+3} - \frac{10}{3}p^{2k+2} + \frac{5}{3}p^{2k+1} + \frac{5}{3}p^{k+3} + \\
&\quad \frac{5}{6}p^{k+2} - \frac{5}{6}p^{k+1}.
\end{aligned}$$

Note that

$$\begin{aligned}
M_2\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &= \frac{1}{2}p^{4k+1} + \frac{5}{2}p^{3k+2} - 6p^{3k+1} + \frac{1}{2}p^{2k+4} - \frac{3}{2}p^{2k+3} - \frac{23}{2}p^{2k+2} + \\
&\quad \frac{43}{2}p^{2k+1} - \frac{9}{2}p^{k+4} + 16p^{k+3} + \frac{7}{2}p^{k+2} - 25p^{k+1} + 7p^4 - 26p^3 + \\
&\quad 19p^2 + 4p \\
&= \left(\frac{5}{2}p^{3k+2} - \frac{5}{6}p^{3k+1} - \frac{5}{3}p^{2k+3} - \frac{10}{3}p^{2k+2} + \frac{5}{3}p^{2k+1} + \frac{5}{3}p^{k+3} + \right. \\
&\quad \left. \frac{5}{6}p^{k+2} - \frac{5}{6}p^{k+1}\right) + \left(\frac{1}{2}p^{2k+4} - \frac{1}{6}p^{2k+3} - \frac{1}{3}p^{k+5} - \frac{2}{3}p^{k+4} + \right. \\
&\quad \left. \frac{1}{3}p^{k+3} + \frac{1}{3}p^5 + \frac{1}{6}p^4 - \frac{1}{6}p^3\right) + \left(\frac{1}{3}p^{2k+3} - \frac{1}{9}p^{2k+2} - \frac{2}{9}p^{k+4} - \right. \\
&\quad \left. \frac{4}{9}p^{k+3} + \frac{2}{9}p^{k+2} + \frac{2}{9}p^4 + \frac{1}{9}p^3 - \frac{1}{9}p^2\right) + \left(-\frac{145}{18}p^{2k+2} + \right. \\
&\quad \frac{145}{54}p^{2k+1} + \frac{145}{27}p^{k+3} + \frac{290}{27}p^{k+2} - \frac{145}{27}p^{k+1} - \frac{145}{27}p^3 - \frac{145}{54}p^2 + \\
&\quad \left. \frac{145}{54}p\right) + \frac{1}{2}p^{4k+1} - \frac{31}{6}p^{3k+1} + \frac{463}{27}p^{2k+1} + \frac{1}{3}p^{k+5} - \frac{65}{18}p^{k+4} + \\
&\quad \frac{245}{27}p^{k+3} - \frac{224}{27}p^{k+2} - \frac{1015}{54}p^{k+1} - \frac{1}{3}p^5 + \frac{119}{18}p^4 - \frac{1111}{54}p^3 + \\
&\quad \frac{1177}{54}p^2 + \frac{71}{54}p \\
&= \left(\frac{5}{3}p^k + \frac{1}{3}p^2 + \frac{2}{9}p - \frac{145}{27}\right)M_2\left(\Gamma_{\mathbb{Z}_n}\right) + \frac{1}{2}p^{4k+1} - \frac{31}{6}p^{3k+1} + \\
&\quad \frac{463}{27}p^{2k+1} + \frac{1}{3}p^{k+5} - \frac{65}{18}p^{k+4} + \frac{245}{27}p^{k+3} - \frac{224}{27}p^{k+2} - \\
&\quad \frac{1015}{54}p^{k+1} - \frac{1}{3}p^5 + \frac{119}{18}p^4 - \frac{1111}{54}p^3 + \frac{1177}{54}p^2 + \frac{71}{54}p.
\end{aligned}$$

Let $X = p^k$, then $\frac{1}{2}p^{4k+1} - \frac{31}{6}p^{3k+1} + \frac{463}{27}p^{2k+1} = p^{2k+1}\left(\frac{1}{2}p^{2k} - \frac{31}{6}p^k + \frac{463}{27}\right) = p^{2k+1}\left(\frac{1}{2}X^2 - \frac{31}{6}X + \frac{463}{27}\right)$. Observe that the quadratic expression $\dfrac{1}{2}X^2 - \dfrac{31}{6}X + \dfrac{463}{27}$ has a positive leading coefficient $(a > 0)$ and a negative discriminant $(D < 0)$; hence, it is strictly positive for all real X. Similarly, consider $\frac{119}{18}p^4 - \frac{1111}{54}p^3 + \frac{1177}{54}p^2 = p^2\left(\frac{119}{18}p^2 - \frac{1111}{54}p + \frac{1177}{54}\right)$. Since the corresponding quadratic term also satisfies $a > 0$ and $D < 0$, the expression is positive for prime numbers p. Moreover, the respective minimum values of these expressions are given by $p^{2k+1} \cdot \dfrac{821}{216} = 3,8p^{2k+1}$ and $p^2 \cdot \dfrac{446.435}{77.112} = 5,78p^2$. It follows immediately that $\dfrac{245}{27}p^{k+3} - \dfrac{224}{27}p^{k+2} > 0$ and $\dfrac{1}{3}p^{k+5} > 0$. Therefore,

$$M_2\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) > \left(\frac{5}{3}p^k + \frac{1}{3}p^2 + \frac{2}{9} - \frac{145}{27}\right)M_2\left(\Gamma_{\mathbb{Z}_n}\right) - \frac{65}{18}p^{k+4} - \frac{1015}{54}p^{k+1} - \frac{1}{3}p^5.$$

The subsequent theorem characterizes the hyper-Zagreb index of $L\left(\Gamma_{\mathbb{Z}_n}\right)$.

Theorem 3.8. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the prime coprime graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, then the hyper Zagreb index of $L\left(\Gamma_{\mathbb{Z}_n}\right)$ is*

$$\begin{aligned}
HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) = 2p^{4k+1} &+ 11p^{3k+2} - 25p^{3k+1} + 2p^{2k+4} - 9p^{2k+3} - 45p^{2k+2} + \\
&88p^{2k+1} - 15p^{k+4} + 65p^{k+3} + 11p^{k+2} - 101p^{k+1} - p^5 + 27p^4 \\
&- 103p^3 + 77p^2 + 16p.
\end{aligned}$$

Proof. Observe that

$$HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) = \sum_{v \in V} \sum_{\{e,f\} \subseteq \in E_v} \left(\deg_L(e) + \deg_L(f)\right)^2,$$

where E_v denotes the set of edges in $\Gamma_{\mathbb{Z}_n}$ incident with the vertex v. By employing the same reasoning as in Theorem 3.6, we obtain

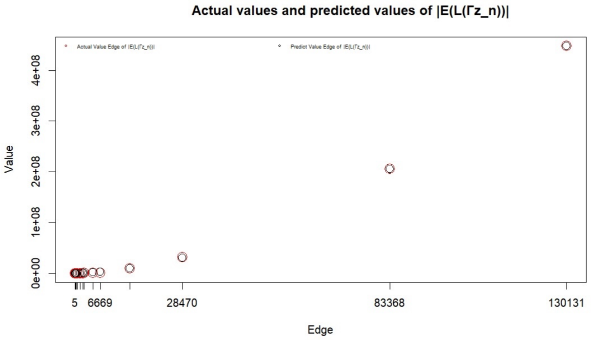

Figure 2. Actual Values and Predicted Values of $\left|E\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right|$

$$\begin{aligned}
HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) =& \left(\frac{1}{8}(p+1)p(p-1)(p-2)\right)(\deg_L((u_1,v_1))+\deg_L((r_1,s_1)))^2\\
&+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)(\deg_L((u_1,v_1))+\deg_L((u_1,v_2)))^2\\
&+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)(\deg_L((u_1,v_1))+\deg_L((u_2,v_1)))^2\\
&+\left(\frac{1}{2}\left(p^k-p-1\right)\left(p^{k+1}-p^2\right)\right)(\deg_L((v_1,v_2))+\deg_L((v_1,u_2)))^2\\
&+\left(\frac{1}{2}(p-1)\left(p^{k+1}-p^2\right)\right)(\deg_L((v_1,v_2))+\deg_L((u_1,v_2)))^2\\
=& \left(\frac{1}{8}(p+1)p(p-1)(p-2)\right)\cdot\left(2\left(p^k-2\right)+2\left(p^k-2\right)\right)^2\\
&+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)\cdot\left(2\left(p^k-2\right)+\left(p^k+p-3\right)\right)^2\\
&+\left(\left(p^k-p\right)\cdot\frac{1}{2}(p-1)p\right)\cdot\left(2\left(p^k-2\right)+\left(p^k+p-3\right)\right)^2\\
&+\left(\frac{1}{2}\left(p^k-p-1\right)\left(p^{k+1}-p^2\right)\right)\left(\left(p^k+p-3\right)+\left(p^k+p-3\right)\right)^2\\
&+\left(\frac{1}{2}(p-1)\left(p^{k+1}-p^2\right)\right)\cdot\left(\left(p^k+p-3\right)+\left(p^k+p-3\right)\right)^2\\
=& 2(p+1)p(p-1)(p-2)\left(p^k-2\right)^2+\left(p^k-p\right)(p-1)p\cdot\\
&\left(3p^k+p-7\right)^2+2\left(p^k-2\right)\left(p^{k+1}-p^2\right)\cdot\left(p^k+p-3\right)^2\\
=& \left(2p^{2k+4}-4p^{2k+3}-2p^{2k+2}+4p^{2k+1}-8p^{k+4}+16p^{k+3}+8p^{k+2}\right.\\
&\left.-16p^{k+1}+8p^4-16p^3-8p^2+16p\right)+\left(9p^{3k+2}-9p^{3k+1}-3p^{2k+3}\right.\\
&-39p^{2k+2}+42p^{2k+1}-5p^{k+4}+33p^{k+3}+21p^{k+2}-49p^{k+1}-\\
&\left.p^5+15p^4-63p^3+49p^2\right)+\left(2p^{4k+1}+2p^{3k+2}-16p^{3k+1}-\right.\\
&2p^{2k+3}-4p^{2k+2}+42p^{2k+1}-2p^{k+4}+16p^{k+3}-18p^{k+2}-\\
&\left.36p^{k+1}+4p^4-24p^3+36p^2\right)\\
HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) =& 2p^{4k+1}+11p^{3k+2}-25p^{3k+1}+2p^{2k+4}-9p^{2k+3}-45p^{2k+2}+\\
&88p^{2k+1}-15p^{k+4}+65p^{k+3}+11p^{k+2}-101p^{k+1}-p^5+27p^4-\\
&103p^3+77p^2+16p.
\end{aligned}$$

Theorem 3.9. *Let $L\left(\Gamma_{\mathbb{Z}_n}\right)$ be the prime coprime line graph of the $\mathbb{Z}_n$ group. If $n = p^k$ with p prime and integer $k \geq 2$, the relation between the hyper Zagreb index of the prime coprime graph and that of its line graph is given by*

$$HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) < \left(2p^k+5p-17\right)HM\left(\Gamma_{\mathbb{Z}_n}\right)+14p^{2k+4}.$$

Proof. According to Theorem 2.5, we have $HM\left(\Gamma_{\mathbb{Z}_n}\right) = p^{3k+1}+3p^{2k+2}-4p^{2k+1}-p^{k+3}-4p^{k+2}+5p^{k+1}-p^4+2p^3+p^2-2p$. Consequently, it follows that

$$\begin{aligned}
2p^kHM\left(\Gamma_{\mathbb{Z}_n}\right) =& 2p^{4k+1}+6p^{3k+2}-8p^{3k+1}-2p^{2k+3}-8p^{2k+2}+10p^{2k+1}-\\
&2p^{k+4}+4p^{k+3}+2p^{k+2}-4p^{k+1}\\
5pHM\left(\Gamma_{\mathbb{Z}_n}\right) =& 5p^{3k+2}+15p^{2k+3}-20p^{2k+2}-5p^{k+4}-20p^{k+3}+25p^{k+2}-\\
&5p^5+10p^4+5p^3-10p^2\\
-17HM\left(\Gamma_{\mathbb{Z}_n}\right) =& -17p^{3k+1}-51p^{2k+2}+68p^{2k+1}+17p^{k+3}+68p^{k+2}\\
&-85p^{k+1}+17p^4-34p^3-17p^2+34p.
\end{aligned}$$

Thus,

$$\begin{aligned}
HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) =& 2p^{4k+1}+11p^{3k+2}-25p^{3k+1}+2p^{2k+4}-9p^{2k+3}-45p^{2k+2}+\\
&88p^{2k+1}-15p^{k+4}+65p^{k+3}+11p^{k+2}-101p^{k+1}-p^5+27p^4-\\
&103p^3+77p^2+16p\\
=& \left(2p^{4k+1}+6p^{3k+2}-8p^{3k+1}-2p^{2k+3}-8p^{2k+2}+10p^{2k+1}-\right.\\
&\left.2p^{k+4}+4p^{k+3}+2p^{k+2}-4p^{k+1}\right)+\left(5p^{3k+2}+15p^{2k+3}-\right.\\
&\left.20p^{2k+2}-5p^{k+4}-20p^{k+3}+25p^{k+2}-5p^5+10p^4+5p^3-10p^2\right)\\
&+\left(-17p^{3k+1}-51p^{2k+2}+68p^{2k+1}+17p^{k+3}+68p^{k+2}-\right.\\
&\left.85p^{k+1}+17p^4-34p^3-17p^2+34p\right)+10p^{2k+1}-8p^{k+4}+\\
&64p^{k+3}-84p^{k+2}-12p^{k+1}+4p^5-74p^3+104p^2-18p\\
=& 2p^kHM\left(\Gamma_{\mathbb{Z}_n}\right)+5pHM\left(\Gamma_{\mathbb{Z}_n}\right)-17HM\left(\Gamma_{\mathbb{Z}_n}\right)+2p^{2k+4}-\\
&22p^{2k+3}+34p^{2k+2}+10p^{2k+1}-8p^{k+4}+64p^{k+3}-84p^{k+2}-\\
&12p^{k+1}+4p^5-74p^3+104p^2-18p\\
=& \left(2p^k+5p-17\right)HM\left(\Gamma_{\mathbb{Z}_n}\right)+2p^{2k+4}-22p^{2k+3}+34p^{2k+2}+\\
&10p^{2k+1}-8p^{k+4}+64p^{k+3}-84p^{k+2}-12p^{k+1}+4p^5-74p^3+\\
&104p^2-18p.
\end{aligned}$$

To ensure certain quadratic forms have a negative discriminant $D < 0$, we adjust coefficients to obtain a sharper upper bound. Observe that, for p prime and integer $k \geq 2$,

$$\begin{aligned}
-22p^{2k+3}+34p^{2k+2}+10p^{2k+1} &< -10p^{2k+3}+34p^{2k+2}-29p^{2k+1}\\
&= p^{2k+1}\left(-10p^2+34p-29\right),
\end{aligned} \tag{4}$$

and

$$\begin{aligned}
-74p^3+104p^2-18p &< -50p^3+104p^2-55p\\
&= p\left(-50p^2+104p-55\right)
\end{aligned} \tag{5}$$

Consider the quadratics $-10p^2+34p-29$ and $-50p^2+104p-55$. Both have negative leading coefficients and negative discriminants, $a < 0$ and $D < 0$. Hence, each quadratic is strictly negative for primes p. In particular, since the right-hand sides of 4 and 5 are positive powers of p multiplied by these quadratics, it follows that 4 and 5 are negative. Consequently, we obtain

$$\begin{aligned}
HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) <& \left(2p^k+5p-17\right)HM\left(\Gamma_{\mathbb{Z}_n}\right)+4p^{2k+4}+\left(-10p^{2k+3}+34p^{2k+2}-\right.\\
&\left.29p^{2k+1}\right)-8p^{k+4}+64p^{k+3}-84p^{k+2}+28p^{k+1}+4p^5+\\
&\left(-50p^3+104p^2-55p\right)\\
<& \left(2p^k+5p-17\right)HM\left(\Gamma_{\mathbb{Z}_n}\right)+4p^{2k+4}-8p^{k+4}+64p^{k+3}-\\
&84p^{k+2}+28p^{k+1}+4p^5\\
<& \left(2p^k+5p-17\right)HM\left(\Gamma_{\mathbb{Z}_n}\right)+4p^{2k+4}+64p^{k+3}+28p^{k+1}+4p^5\\
=& \left(2p^k+5p-17\right)HM\left(\Gamma_{\mathbb{Z}_n}\right)+4p^{2k+4}\left(1+16p^{-(k+1)}+\right.\\
&\left.7p^{-(k+3)}+p^{-(2k-1)}\right).
\end{aligned}$$

Since

$$
\begin{aligned}
p^{-(k+1)} &= \frac{1}{p^{k+1}} < \frac{1}{2^{2+1}} = \frac{1}{8},\\
p^{-(k+3)} &= \frac{1}{p^{k+3}} < \frac{1}{2^{2+3}} = \frac{1}{32},\\
p^{-(2k-1)} &= \frac{1}{p^{2k-1}} < \frac{1}{2^{2\cdot 2-1}} = \frac{1}{8},
\end{aligned}
$$

then

$$
\begin{aligned}
HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &< \left(2p^k + 5p - 17\right) HM\left(\Gamma_{\mathbb{Z}_n}\right) + 4p^{2k+4}\left(1 + 16\cdot\frac{1}{8} + 7\cdot\frac{1}{32} + \frac{1}{8}\right)\\
&= \left(2p^k + 5p - 17\right) HM\left(\Gamma_{\mathbb{Z}_n}\right) + 4p^{2k+4}\cdot\frac{107}{11}\\
HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) &< \left(2p^k + 5p - 17\right) HM\left(\Gamma_{\mathbb{Z}_n}\right) + 14p^{2k+4}.
\end{aligned}
$$

The following parts presents the statistical analysis of the obtained results. Regression modeling with a non-linear approach was performed to examine the relationships between: (i) the number of edges of $\Gamma_{\mathbb{Z}_n}$ and the number of edges of $L\left(\Gamma_{\mathbb{Z}_n}\right)$, (ii) the number of vertices of $\Gamma_{\mathbb{Z}_n}$ and the number of vertices of $L\left(\Gamma_{\mathbb{Z}_n}\right)$, and (iii) the hyper-Zagreb index of $\Gamma_{\mathbb{Z}_n}$ and the hyper-Zagreb index of $L\left(\Gamma_{\mathbb{Z}_n}\right)$.

The modeling process was carried out using the R-statistical software package. Based on the regression analysis, the following non-linear models are obtained:

For the relationship between the number of edges of $\Gamma_{\mathbb{Z}_n}$ and the number of edges of $L\left(\Gamma_{\mathbb{Z}_n}\right)$ according to Theorem 2.2 and Theorem 3.3, the fitted model is

$$\left|E\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right| = 231.6\times\left|E\left(\Gamma_{\mathbb{Z}_n}\right)\right| + 0.0375\times\left|E\left(\Gamma_{\mathbb{Z}_n}\right)\right|^2 - 4.674\times 10^{-8}\times\left|E\left(\Gamma_{\mathbb{Z}_n}\right)\right|^3.$$

For the relationship between the number of vertices of $\Gamma_{\mathbb{Z}_n}$ which is p^k and the number of vertices of $L\left(\Gamma_{\mathbb{Z}_n}\right)$ which is equal to the number of edges of $\Gamma_{\mathbb{Z}_n}$ as given in Theorem 2.2, the regression model is

$$\left|V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right| = 10.58\times\left|V\left(\Gamma_{\mathbb{Z}_n}\right)\right| + 0.001237\times\left|V\left(\Gamma_{\mathbb{Z}_n}\right)\right|^2.$$

For the relationship between the hyper Zagreb index of $\Gamma_{\mathbb{Z}_n}$ and that of $L\left(\Gamma_{\mathbb{Z}_n}\right)$ according to Theorem 2.5 and Theorem 3.8, the model is given by

$$HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right) = 5.767\times 10^{-5}\times HM\left(\Gamma_{\mathbb{Z}_n}\right)^2 - 9.431\times 10^{-16}\times HM\left(\Gamma_{\mathbb{Z}_n}\right)^3.$$

Based on the constructed regression models, comparison plots are generated to evaluate the agreement between the observed values and the predicted values of the parameters of the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$. The statistical analysis is conducted using a dataset obtained from the computed structural characteristics of $\Gamma_{\mathbb{Z}_n}$ for different values of the graph order n. For each value of n, several structural parameters of $\Gamma_{\mathbb{Z}_n}$, including the number of vertices, the number of edges, and the hyper-Zagreb index, are determined analytically and used as explanatory variables to estimate the corresponding parameters of the associated line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$.

To model the relationship between these parameters, non-linear regression models are employed. The use of non-linear regression is motivated by the non-linear patterns observed in the computed dataset, where the structural parameters of the line graph do not increase proportionally with those of the original graph.

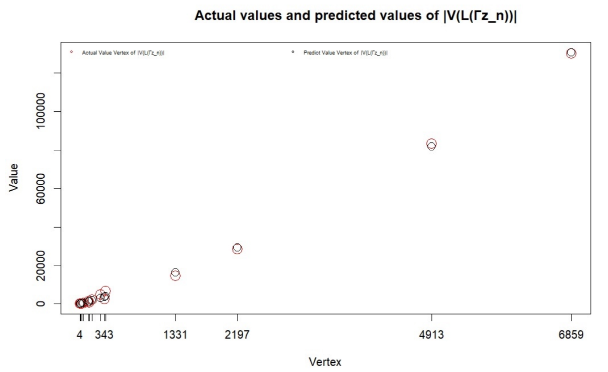

Figure 3. Actual Values and Predicted Values of $\left|V\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)\right|$

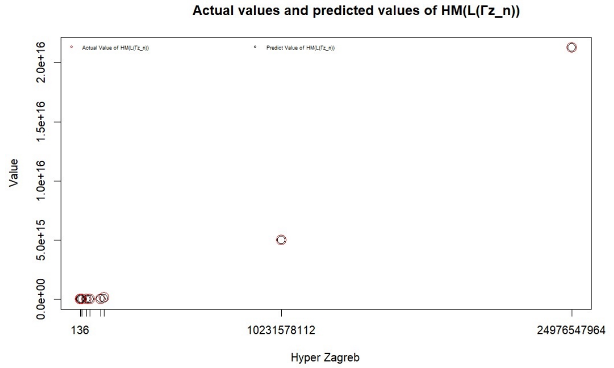

Figure 4. Actual Values and Predicted Values of $HM\left(L\left(\Gamma_{\mathbb{Z}_n}\right)\right)$

Figures 2, 3, and 4 illustrate the comparison between the observed values and those predicted by the fitted regression models. In each figure, the plotted points are concentrated near the identity line, indicating strong agreement between the predicted and actual values. The result suggests that the proposed regression models effectively approximate the functional relationship between the structural parameters of $\Gamma_{\mathbb{Z}_n}$ and those of its line graph.

From a theoretical perspective, the observed behavior of the Zagreb-type indices can be explained by the structural properties of line graphs. In the line graph $L\left(\Gamma_{\mathbb{Z}_n}\right)$, each vertex corresponds to an edge of the original graph, and adjacency between vertices reflects the incidence of edges in $\Gamma_{\mathbb{Z}_n}$. Consequently, the degree of a vertex in the line graph depends on the degrees of the endpoints of the corresponding edge in the original graph. This transformation modifies the vertex-degree distribution while preserving local adjacency patterns, which directly affects the magnitude of degree-based topological indices such as the hyper-Zagreb index.

The regression analysis further reveals that these indices exhibit a stable functional relationship with respect to the graph order n. As the size of the graph increases, the structural parameters of $L\left(\Gamma_{\mathbb{Z}_n}\right)$ follow consistent growth patterns that reflect the underlying algebraic structure of $\Gamma_{\mathbb{Z}_n}$. Such regular behavior indicates that the structural descriptors of line graphs can be effectively estimated from the corresponding parameters of the original graph through suitable non-linear models.

These findings have potential implications both in graph theory in QSPR modeling. In graph theory, the results provide insight into how graph transformations, particularly the line-graph operation, influence degree-based topological indices. In QSPR studies, where chemical compounds are often represented by molecular graph, reliable relationships between graph invariants and structural parameters are essential for predicting physicochemical and biological properties of molecules. Therefore, the proposed regression framework may provide a useful tool for estimating molecular descriptors associated with line-graph transformations in chemical graph theory (Mahboob et al., 2024; Gayathri and Roy, 2025; Hakeem et al., 2025).

4. CONCLUSIONS

This paper presented a comprehensive investigation of the line graph of the prime coprime graph associated with the integers modulo group. Explicit general formulas for the first Zagreb index, second Zagreb index, and hyper-Zagreb index were derived for the considered structures. A comparison with previously reported Zagreb indices of the prime coprime graph shows how the line graph transformation affects the degree-based topological descriptors and their growth behavior. In addition to the theoretical derivations, a statistical analysis was conducted to explore the quantitative relationships between the prime coprime graph and its corresponding line graph. The findings demonstrated strong relations among the computed Zagreb-based indices, highlighting their complementary roles in capturing both local degree interactions and global structural complexity. These results contribute to a deeper understanding of algebraically defined graphs from both combinatorial and statistical perspectives. For future research, the proposed framework may be extended to other families of topological indices. Moreover, further investigations may consider line graphs derived from non-abelian groups, composite algebraic structures, or other algebraically generated graph classes, thereby broadening the applicability of Zagreb-based analysis in algebraic graph theory and quantitative structure analysis.

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REFERENCES

Abdurahim, A., M. U. Romdhini, and S. K. S. Husain (2025). Zagreb-Based Indices of Prime Coprime Graph for Integers Modulo Power of Primes. *Malaysian Journal of Mathematical Sciences*, **19**(3); 961–974

Abdurahim, A., M. U. Romdhini, J. Qudsi, F. Al-Sharqi, and Z. Md Rodzi (2026). Delta Degree-Based Indices of Prime Coprime Graph for Integers Modulo Group. *Science and Technology Indonesia*, **11**(1); 10–18

Abel, L. A., D. Welyyanti, L. Yulianti, and D. Permana (2025). The Locating Chromatic Number of the Cyclic Chain Graph. *Science and Technology Indonesia*, **10**(3); 958–962

Adhikari, A. and S. S. Banerjee (2021). Prime Coprime Graph of a Finite Group. *Novi Sad Journal of Mathematics*, **52**; 11151

Ahmed, W. E., M. F. Hanif, M. Hussain, and M. K. Siddiqui (2025). A Comprehensive Study on Topological Indices and Entropy Measures for Terbium Niobate Using Logarithmic Regression Models. *Computational Biology and Chemistry*, **119**; 108558

Arockiaraj, M., D. Paul, J. Clement, S. Tigga, K. Jacob, and K. Balasubramanian (2023). Expected Value of First Zagreb Connection Index in Random Cyclooctatetraene Chain, Random Polyphenyls Chain, and Random Chain Network. *SAR and QSAR in Environmental Research*, **34**; 569–589

Chang, S., J. Li, and Y. Zheng (2024). The Eigenvalue Multiplicity of Line Graphs. *Linear Algebra and its Applications*, **703**; 47–62

Chen, X. (2023). General Sum-Connectivity Index of a Graph and Its Line Graph. *Applied Mathematics and Computation*, **443**; 127779

Das, K. C., K. Xu, and J. Nam (2015). Zagreb Indices of Graphs. *Frontiers of Mathematics in China*, **10**(3); 567–582

Durgut, R. and T. Turacı (2023). Computing Some Eccentricity-Based Topological Indices of Para-Line Graphs of Hexagonal Cactus Chains. *Polycyclic Aromatic Compounds*, **43**(1); 115–130

Estrada, E. (2022). Statistical–Mechanical Theory of Topological Indices. *Physica A: Statistical Mechanics and its Applications*, **602**; 127612

Fitriani, F., I. E. Wijayanti, A. Faisol, and S. Ali (2025). Commuting and Centralizing Maps on Modules. *Science and Technology Indonesia*, **10**(3); 690–697

Gayathri, K. B. and S. Roy (2025). Quantitative Structure Property Relationship Study of Postpartum Depression Medications Using Topological Indices and Regression Models. *Ain Shams Engineering Journal*, **16**(1); 103194

Gonzalez-Diaz, H., S. Vilar, L. Santana, and E. Uriarte (2007). Medicinal Chemistry and Bioinformatics - Current Trends in Drugs Discovery with Networks Topological Indices. *Current Topics in Medicinal Chemistry*, **7**(10); 569–589

Gutman, I. and K. C. Das (2004). The First Zagreb Index 30 Years After. *MATCH Communications in Mathematical and in Computer Chemistry*, **50**; 83–92

Hakeem, A., A. Ullah, S. Zaman, E. E. Mahmoud, H. Ahmad,

P. Ali, and M. B. Belay (2025). Topological Modeling and QSPR Based Prediction of Physicochemical Properties of Bioactive Polyphenols. *Scientific Reports*, **15**(27466); 1–10

Hassan, M. M. (2023). Connection Number-based Multiplicative Zagreb Indices of Chemical Structures. *Current Organic Chemistry*, **27**(21); 1873–1881

Islam, S. R., B. Mohsin, and M. Pal (2024). Hyper-Zagreb Index in Fuzzy Environment and Its Application. *Heliyon*, **10**(16); e36110

Li, G. and M. Zhang (2022). Sharp Bounds on the Arithmetic-Geometric Index of Graphs and Line Graphs. *Discrete Applied Mathematics*, **318**; 47–60

Mahboob, A., M. W. Rasheed, A. M. Dhiaa, I. Hanif, and L. Amin (2024). On Quantitative Structure-Property Relationship (QSPR) Analysis of Physicochemical Properties and Anti-Hepatitis Prescription Drugs Using a Linear Regression Model. *Heliyon*, **10**(4); e25908

Mannan, M. A., H. Akter, and M. A. Ullah (2022). Evaluate All The Order of Every Element in The Higher Even, Odd, and Prime Order of Group for Composition. *Science and Technology Indonesia*, **7**(3); 333–343

Manonmani, R. and P. Selvarani (2025). Multilinear Regression Analysis of Antiviral Medications with Topological Indices. *Journal of Molecular Graphics and Modelling*, **140**; 109104

Noureen, S., A. Ali, A. A. Bhatti, A. M. Alanazi, and Y. Shang (2024). Predicting Enthalpy of Formation of Benzenoid Hydrocarbons and Ordering Molecular Trees Using General Multiplicative Zagreb Indices. *Heliyon*, **10**(10); e30913

Pongpipat, D. and N. Nupo (2026). On Vertex-Degree-Based Topological Indices of Cayley Digraphs of Rectangular Groups and Their Role as Molecular Descriptors. *Science and Technology Indonesia*, **11**(2); 732–741

Raza, Z., S. Akhter, and Y. Shang (2023). Expected Value of First Zagreb Connection Index in Random Cyclooctatetraene Chain, Random Polyphenyls Chain, and Random Chain Network. *Frontiers in Chemistry*, **10**; 1067874

Romdhini, M. U., A. Abdurahim, A. E. S. H. Maharani, and S. R. Kamali (2025). Transmission-Based Energies of Prime Coprime Graph for Integers Modulo Group. *Science and Technology Indonesia*, **10**(3); 779–785

Rosary, M. S. (2022). Topological Study of Line Graph of Remdesivir Compound Used in the Treatment of Corona Virus. *Polycyclic Aromatic Compounds*, **42**(8); 5731–5747

Shirdel, G. H., H. Rezapour, and A. M. Sayadi (2013). The Hyper-Zagreb Index of Graph Operations. *Iranian Journal of Mathematical Chemistry*, **4**(2); 213–220

Ul Ain, H., R. Ismail, M. K. Jamil, M. E. M. Abdulla, and M. Azeem (2024). Extremal Graph of Super Line Graph Operation Via Generalized Randić Index. *Ain Shams Engineering Journal*, **15**(10); 102938

Usman, M. and M. Javaid (2024). Connection-Based Zagreb Indices of Polycyclic Aromatic Hydrocarbons Structures. *Current Organic Synthesis*, **21**(3); 246–256

Yow, K. S., M. A. Javed, A. Q. Baig, M. Ahmad, S. Ghareeb, and A. Qayyum (2026). Computational Analysis of Non-Kekule Benzenoid Molecules for Medical Applications. *Malaysian Journal of Mathematical Sciences*, **20**(1); 199–211

Zhen, W., D. Wong, and S. Xu (2025). Line Graphs with the Largest Eigenvalue Multiplicity. *Discrete Mathematics*, **348**(9); 114562