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Research Paper

The Relation of Tribinomial Coefficients with Triangular, Catalan and Mersenne Number

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Abstract
Numbers are inseparable from mathematics. Each of these types of numbers has its own distinct definition and properties. Numbers are not only used in mathematics, but are also essential in other fields such as philosophy, technology, and science. Tribinomial coefficients, Catalan numbers, and Mersenne numbers are three types of numbers that has their own uniqueness and beauty. Tribinomial coefficients derived from triangular number using similar definition for binomial coefficients. Triangular numbers constitute a class of figurative numbers derived from the systematic arrangement of discrete units (such as dots) into the geometric configuration of an equilateral triangle. The Catalan numbers constitute a sequence of positive integers that emerge in numerous combinatorial enumeration problems. Formally, the n-th Catalan number is defined by the closed-form expression: $C_n = \frac{1}{n+1}\binom{2n}{n} = \frac{(2n)!}{(n+1)!n!}, n \in N$. Mersenne numbers are the numbers known in mathematics that also have their own beauty and uniqueness. When represented using binary, all Mersenne numbers are repeating 1s. The first eight Mersenne numbers are 1, 3, 7, 15, 31, 63, 127, 255, which are represented in binary as 1, 11, 111, 1111, 11111, 111111, 1111111, 11111111. In this study the relationship of Tribinomial coefficients with Catalan and Marsenne numbers will be disccused.

Keywords
Enumeration, Tribinomial Coefficients, Triangular Numbers, Catalan Numbers, Mersenne Number

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1. INTRODUCTION

Although not as well-known as the binomial coefficients, the tribinomial coefficients have their own uniqueness and beauty. This number is derived from triangular numbers. A triangular number is defined as the total number of discrete elements that can be arranged to form an equilateral triangular configuration. These numbers are conventionally denoted by $T_n = \frac{n(n+1)}{2}$, $n \in N$, where n specifies the number of points positioned along each side of the triangle (Chen and Fang, 2007). Using the same idea for defining binomial coefficients but applied to triangular numbers, the Tribinomial coefficients are defined as $\begin{bmatrix} n \\ r \end{bmatrix} = \frac{t_n^*}{t_r^* t_{n-r}^*}$, where $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1$, $0 \leq r \leq n$, and $t_n^* = T_n T_{n-1} \cdots T_2 T_1$. The definition of t_n^* is analogous to that of the factorial function $n! = n(n-1)\cdots 2 \cdot 1$ (Koshy, 2009).

Catalan numbers are a type of number known in mathematics, named in honor of Eugene Catalan, a Belgian scientist, after he studied the valid patterns of parentheses sequences Pak (2014). Catalan numbers admit several equivalent formulations, one of which arises from the triangulations of convex polygons. Specifically, the nth Catalan number counts the number of distinct ways a convex polygon with $n+2$ sides can be partitioned into non-overlapping triangles by drawing non-intersecting diagonals. Some researchers had investigated these triangular forms of the convex polygon including Cayley (1891) and Selimi and Saračević (2019). Another geometric representation of Catalan numbers is lattice path which have been investigated by several researchers, including Breckenridge et al. (1991), Bayer and Brandt (2015), Roman (2015), Stanley (2015),Saracević et al. (2017); Saračević et al. (2018), Chu (2018), and Armstrong (2024). Wamiliana et al. (2023) investigate the relationship of multiset and Catalan Numbers, and Amansyah et al. (2024) investigated the noncrossing partition of odd and even numbers with Catalan Numbers. Catalan numbers also applied in many areas such as for data security in computer science, for example in the research conducted by Saračević et al. (2019, 2021), and Mukhammadovich and Djuraevich (2023). Catalan numbers also applied to determine the combinatorial form of the secondary structure of RNA Hofacker et al. (1998), Alexiou et al. (2011), Ndagijimana (2016), and many other applications.

A Mersenne number is a number in the form of $M = 2^n - 1, n \geq 1$. The sequence of these numbers is growing exponentially, and recently the biggest Mersenne Number found is $2^{136,279,841}$-1, having 41,024,320 decimal digits using GIMPS (Great Internet Mersenne Prime Search) by Luke Durant in October 12th, 2024 (https://www.mersenne.org/primes/?press=M136279841). In this study the relation among Tribinomial coefficients, Triangular Numbers, Catalan Numbers, and Mersenne Numbers, will be discussed.

2. LITERATURE REVIEW

2.1 Triangular Numbers

Triangular numbers are defined as the class of figurate numbers generated by arranging discrete objects in the geometric form of an equilateral triangle. Formally, they are denoted by T_n, where n specifies the number of points constituting each side of the triangular configuration (Chen and Fang, 2007). The Equation (1) is the triangular numbers.

$$T_n = \frac{n(n+1)}{2}, n \in N \tag{1}$$

Figure 1 shows some Triangular Numbers. The nth Triangular number is defined as the total count of points arranged in a triangular configuration with n points along each side. Equivalently, it can be expressed as the sum of the first n positive integers.

2.2 Binomial Coefficients

Suppose n and r are non-negative integers, the binomial coefficient $\binom{n}{r}$ is defined as: $\binom{n}{r} = \frac{n!}{r!(n-r)!}$ where $n! = n(n-1)(n-2)\cdots(2)(1)$, and $0 \le r \le n$. If $r > n$ then $\binom{n}{r}$ is defined as 0 (Koshy, 2009). Equation (2), Equation (3), and Equation (4) are some identities related with Binomial Coefficients according to Graham et al. (1994).

$$\binom{r}{k} = \frac{r}{k}\binom{r-1}{k-1}, k \neq 0 \tag{2}$$

$$\binom{n}{k} = \binom{n}{n-k}, n \geq 0; n, k \in Z \tag{3}$$

$$\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}, m, n, k, r \in Z \tag{4}$$

2.3 Central Binomial Coefficient

The Central Binomial Coefficient (CBC) denoted by $\binom{2n}{n}$, is located in the middle of the even-numbered row of Pascal's triangle. Figure 2 displays the Central Binomial Coeffiecients (the numbers which are surrounding by circles).

2.4 Tribinomial Coefficients

Using the same idea as the binomial coefficient, the binomial coefficient for triangular numbers, called the tribinomial coefficient $\begin{bmatrix} n \\ r \end{bmatrix}$, is defined in Equation (5) as follows :

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{t_n^*}{t_r^* t_{n-r}^*} \tag{5}$$

where $\begin{bmatrix} n \\ 0 \end{bmatrix} = \begin{bmatrix} n \\ n \end{bmatrix} = 1, 0 \le r \le n$ and $T_n^* = T_n T_{n-1} \cdots T_2 T_1$. The definition of t_n^* analogous to that of the factorial function $n! = n(n-1)\cdots 2 \cdot 1$, and (Koshy, 2009). Moreover, Koshy (2009) stated that the tribinomial coefficient can be defined in Equation (6) and also has the recursive equation as shown in Equation (7).

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{1}{n-r+1}\binom{n+1}{r+1}\binom{n}{r} \tag{6}$$

$$\begin{bmatrix} n \\ r \end{bmatrix} = \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}\frac{T_n}{T_r} = \begin{bmatrix} n-1 \\ r \end{bmatrix}\frac{T_n}{T_{n-r}} \tag{7}$$

2.5 Central Tribinomial Coefficients

Analog with Binomial coefficients that can be arranged and formed Pascal's Triangle, the Tribinomial's Triangle also can be constructed. By using Equation (2) we know that $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix} = \cdots = \begin{bmatrix} n \\ 0 \end{bmatrix} = 1$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} \cdots = \begin{bmatrix} n \\ n \end{bmatrix} = 1$. $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2-1+1}\binom{3}{2}\binom{2}{1} = \frac{1}{2}(3)(2) = 3$; $\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{1}{3-1+1}\binom{4}{2}\binom{3}{1} = \frac{1}{3}(6)(3) = 6$; $\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \frac{1}{3-2+1}\binom{4}{3}\binom{3}{2} = \frac{1}{2}(4)(3) = 6$, and so on. Moreover, using the similar construction of Pascal's Triangle, we get the Tribinomial's Triangle. Moreover, again, analog with Central Binomial Coefficients, the Central Tribinomial Coefficients defined. Figure 2 shows the Tribinomial's Triangle Central Tribinomial Coefficients.

Koshy (2009) stated that the Central Tribinomial Coefficients (CTC) is denoted by $\begin{bmatrix} 2n \\ n \end{bmatrix}$ defined as in Equation (8) as follows:

$$\begin{aligned} \begin{bmatrix} 2n \\ n \end{bmatrix} &= \frac{1}{2n-n+1}\binom{2n+1}{n+1}\binom{2n}{n} \\ &= (2n+1)\cdot\frac{1}{n+1}\binom{2n}{n}\cdot\frac{1}{n+1}\binom{2n}{n} \\ &= (2n+1)(C_n)^2 \end{aligned} \tag{8}$$

where C_n is the n^{th} Catalan number.

2.6 Catalan Numbers

Multiple definitions exist for Catalan numbers, yet the canonical approach, frequently regarded as the most prominent is stated in Equation (9) as follows:

$$C_n = \frac{1}{n+1}\binom{2n}{n} \text{ where } n \geq 0, n \in \mathbb{Z}^+ \tag{9}$$

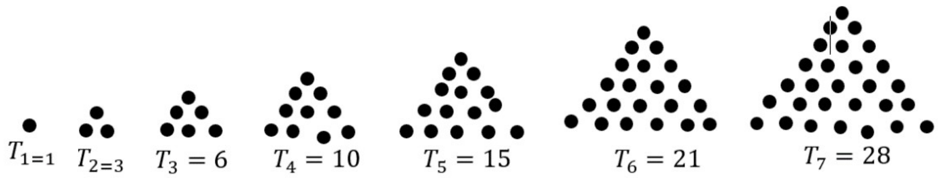

Figure 1. Some Example of Triangular Numbers

Table 1. Tribinomial Coefficients when $0 \leq n \leq 25, 0 \leq r \leq 9$

r\n	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	3	1							
3	1	6	6	1						
4	1	10	20	10	1					
5	1	15	50	50	15	1				
6	1	21	105	175	105	21	1			
7	1	28	196	490	490	196	28	1		
8	1	36	336	1176	1764	1176	336	36	1	
9	1	45	540	2520	5292	5292	2520	540	45	1
10	1	55	825	4950	13860	19404	13860	4950	825	55
11	1	66	1210	9075	32670	60984	60984	32670	9075	1210
12	1	78	1716	15730	70785	169884	226512	169884	70785	15730
13	1	91	2366	26026	143143	429429	736164	736164	429429	143143
14	1	105	3185	41405	273273	1002001	2147145	2760615	2147145	1002001
15	1	120	4200	63700	496860	2186184	5725720	9202050	9202050	5725720

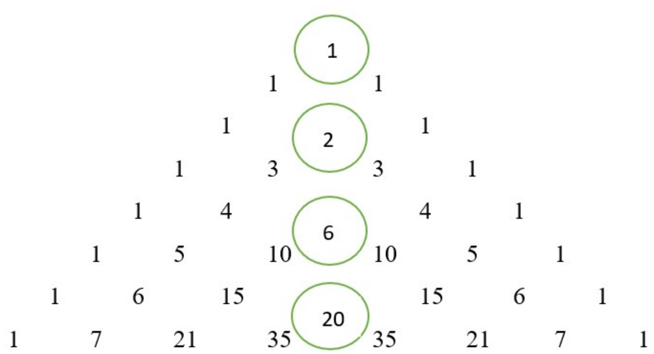

Figure 2. Pascal 's Triangle and Central Binomial Coefficients

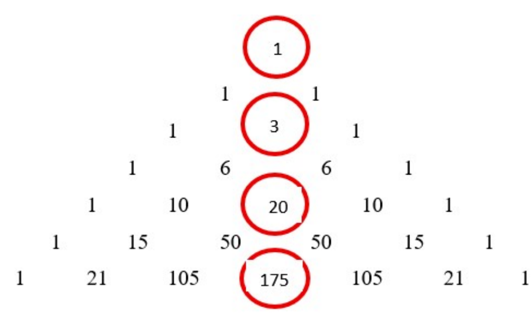

Figure 3. Tribinomial's Triangle and Central Tribinomial Coefficients

2.7 Mersenne Numbers

According to Deza (2022), Mersenne Numbers are are integers of the form:

$$M_p = 2^p - 1, p \in N \tag{10}$$

3. RESULT AND DISCUSSIONS

To see the relationship of Tribinomial coefficients with Catalan Numbers we construct the table that contains Tribinomial coefficients when $0 \leq n \leq 25, 0 \leq r \leq 25$. However, due to the space limitation, we divide the construction into $0 \leq n \leq 25, 0 \leq r \leq 9$; $10 \leq n \leq 25, 0 \leq r \leq 16$; and $0 \leq n \leq 25, 17 \leq r \leq 25$. After calculating the Tribinomial coefficients, then those coefficients are put in tables. Table 1 show the Tribinomial coefficients for $0 \leq n \leq 25, 0 \leq r \leq 9$; Table 2 shows for $10 \leq n \leq 25, 0 \leq r \leq 16$; Table 3 shows for $0 \leq n \leq 25, 17 \leq r \leq 25$. Note that on Table 3, the rightmost column shows the sum of the coefficients in every row.

By using Equation (9), we get the sequence of Catalan Num-

Table 2. Tribinomial Coefficients when $10 \leq n \leq 25$, $10 \leq r \leq 16$

n \ r	10	11	12	13	14	15	16
10	1						
11	66	1					
12	1716	78	1				
13	26026	2366	91	1			
14	273273	41405	3185	105	1		
15	2186184	496860	63700	4200	120	1	
16	14158144	4504864	866320	95200	5440	136	1
17	77364144	32821152	8836464	1456560	138720	6936	153
18	367479684	200443464	71954064	16604784	2372112	197676	8721
19	1551580888	1057896060	488259720	150233760	30046752	3755844	276165
20	5924217936	4936848280	2848181700	112675320	300467520	52581816	5799465
21	20734762776	20734762776	146206666060	722999700	2478857040	578399976	89311761
22	67255063876	79483257308	67255063876	40648664980	17420856420	5226256926	1075994073
23	203982391536	281248448936	281248448936	203982391536	106847919376	40067969766	10606227291
24	582806832960	927192688800	1081724803600	927192688800	582806832960	267119798440	88385227425
25	1578435172060	2869882132000	3863302870000	3863302870000	2869882132000	1578435172600	638337753625

bers as 1,1,2,5,14,42,132,429,1430,4862,16796,58786,2080-12,742900, and so on. Note that these numbers appear in the rightmost column on Table 3. Therefore $\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} = c_{n+1}$.

Theorem 3.1 *Koshy (2009)* $\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} = c_{n+1}$

Proof. By using Binomial Identity in Equation (2) we get

$$\binom{n+1}{n-r+1} = \frac{n+1}{n-r+1}\binom{n}{n-r},$$

$$\frac{1}{n+1}\binom{n+1}{n-r+1} = \frac{1}{n-r+1}\binom{n}{n-r},$$

$$\frac{1}{n+1}\binom{n+1}{n-r+1} = \frac{1}{n-r+1}\binom{n}{r}.$$

Doing the summation for r, $0 \leq r \leq n$ for Tribinomial Equation in Equation (6) we get:

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} = \sum_{r=0}^{n} \frac{1}{n-r+1}\binom{n}{r}\binom{n+1}{r+1}$$

$$= \sum_{r=0}^{n} \frac{1}{n+1}\binom{n+1}{n-r+1}\binom{n+1}{r+1}$$

$$= \frac{1}{n+1}\sum_{r=0}^{n}\binom{n+1}{r+1}\binom{n+1}{n-r+1}$$

$$= \frac{1}{n+1}\sum_{r=0}^{n}\binom{n+1}{1+r}\binom{n+1}{(n+1)-r}$$

$$= \frac{1}{n+1}\binom{(n+1)+(n+1)}{1+(n+1)} \text{ (By Identity in Equation 4)}$$

$$= \frac{1}{n+1}\binom{2n+2}{n+2} = \frac{1}{n+2}\binom{2n+2}{n+1} = C_{n+1}.$$

Table 4 shows the relationship of Tribinomial coefficients with Catalan Numbers (which also shown on the rightmost column shown of Table 3). From Table 4, for $n = 0$, the sum of the Tribinomial coefficients has the as C_1. For $n = 1$, the sum of the Tribinomial coefficients has the same value as C_2. For $n = 2$, the sum of the Tribinomial coefficients has the same value as C_3. For $n = 3$ the sum of the Tribinomial coefficients has the same value as C_3 so on. Thus the n^{th} value of the sum of the Tribinomial coefficients is the same as C_{n+1}.

Theorem 3.2 *(Koshy and Salmassi, 2009)*

The Tribinomial coefficient is odd for every r iff $n = M_k - 1$.

Proof. $\Rightarrow$ From definition of Tribinomial coefficients

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{1}{n-r+1}\binom{n+1}{r+1}\binom{n}{r} = \frac{1}{n-r+1}\frac{n!}{(n-r)!r!}\binom{n+1}{r+1}$$

$$= \frac{n!}{(n-r+1)!r!}\binom{n+1}{r+1} = \frac{1}{n+1}\binom{n+1}{r}\binom{n+1}{r+1}.$$

So

$$\begin{bmatrix} n \\ r+1 \end{bmatrix} = \frac{1}{n+1}\binom{n+1}{r+1}\binom{n+1}{r+2}.$$

Compute the ratio of these two consecutive Tribinomial coefficients:

$$\frac{\begin{bmatrix} n \\ r+1 \end{bmatrix}}{\begin{bmatrix} n \\ r \end{bmatrix}} = \frac{\frac{1}{n-r}\binom{n+1}{r+2}\binom{n}{r+1}}{\frac{1}{n-r+1}\binom{n+1}{r+1}\binom{n}{r}}$$

$$= \frac{(n-r+1)\frac{(n+1)!}{(n-r-1)!(r+2)!}\frac{n!}{(n-r-1)!(r+1)!}}{(n-r)\frac{(n+1)!}{(n-r)!(r+1)!}\frac{n!}{(n-r)!r!}}$$

Table 3. Tribinomial Coefficients when $10 \leq n \leq 25$, $17 \leq r \leq 25$, and the Sum of Coefficients in Every Row

n \ r	17	18	19	20	21	22	23	24	25	Sum
0										1
1										2
2										5
3										14
4										42
5										132
6										429
7										1430
8										4862
9										16796
10										58786
11										208012
12										742900
13										2674440
14										9694845
15										35357670
16										129644790
17	1									477638700
18	171	1								1767263190
19	10830	190	1							6564120420
20	379050	13300	210	1						24466267020
21	8756055	512050	16170	231	1					91482563640
22	147685461	20954865	681835	19481	253	1				343059613650
23	1941008916	238369516	18818646	896126	23276	276	1			1289904147324
24	20796524100	3405278800	376372920	26883780	1163800	27600	300	1		4861946041452
25	187746398125	39525557500	5824819000	582481900	37823500	1495000	32500	325	1	18367353072152

$$= \frac{(n-r+1)(n-r)}{(r+2)(r+1)}.$$

Suppose that $\begin{bmatrix} n \\ r \end{bmatrix}$ is odd for all r, then the ratio $\frac{\begin{bmatrix} n \\ r+1 \end{bmatrix}}{\begin{bmatrix} n \\ r \end{bmatrix}}$ must also odd for all r (otherwise he parity would change somewhere in the row). The ratio $\frac{(n-r+1)(n-r)}{(r+2)(r+1)}$ consists of numerator $(n-r+1)(n-r)$ and denominator $(r+2)(r+1)$. Both of them are the product of two consecutive number. Thus, both numbers have one even and one odd number. As a result, by 2-adic valuation, $v_2(n-r+1)+v_2(n-r) = v_2(r+2)+v_2(r+1)$ for every r, and hence they have the same parity.

Moreover, let $s = r+1$, then $v_2(n-s+2)+v_2(n-s+1) = v_2(s+1)+v_2(s)$, for all s. This symmetry works only if $n+2 = 2^k$. Thus $n = 2^k - 2$. Since $M_k = 2^k - 1$, then $n = (2^k - 1) - 1$, $n = M_k - 1$ Therefore, if the Tribinomial coefficient is odd for every r then $n = M_k - 1$.

$\Leftarrow$ Suppose that $n = M_k - 1$. By definition, the Mersenne number is $M_k = 2^k - 1$. Thus, we get $n = 2^k - 2$ or $n+1 = 2^k - 1$

Substitute $n = 2^k - 2$ to the Tribinomial coefficients we get: $\begin{bmatrix} n \\ r \end{bmatrix} = \frac{1}{2^k-r-1}\binom{2^k-1}{r+1}\binom{2^k-2}{r}$.

From Lucas Theorem, $\binom{2^k-1}{m} \equiv 1 \pmod 2$, for every m because the binary representation of $2^k - 1$ is $(11111\ldots)$. So, every binomial coefficient in that row of Pascal triangle is odd. Thus $\binom{2^k-1}{r+1} \equiv 1 \pmod 2$.

$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{1}{2^k-r-1}\binom{2^k-1}{r+1}\binom{2^k-2}{r}$

Now we examine $\binom{2^2-2}{r}$. Note that the power of 2 in denominator $2^k - r - 1$ cancels the same power in the numerator $\binom{2^k-2}{r}$.

Thus

$$\frac{1}{2^k-r-1}\binom{2^k-1}{r+1}\binom{2^k-2}{r}$$

contains no factor of 2.

Therefore

$$\begin{bmatrix} n \\ r \end{bmatrix} = \frac{1}{2^k-r-1}\binom{2^k-1}{r+1}\binom{2^k-2}{r}$$

is odd.

Thus if $n = M_k - 1$ is a Mersenne number then Tribinomial coefficients is odd.

The Mersenne numbers is of the form $2^p - 1$, then the first eight of Mersenne numbers are 1,3,7,15,31,63,127,255,...

Table 4. The Sum of Tribinomial Coefficients in Every Row and Catalan Numbers

n	The Sum of Every Row of Tribinomial Coefficients	Catalan Numbers
0	1	1
1	2	1
2	5	2
3	14	5
4	42	14
5	132	42
6	429	132
7	1430	429
8	4862	1430
9	16796	4862
10	58786	16796
11	208012	58786
12	742900	208012
13	2674440	742900
14	9694845	2674440
15	35357670	9694845
16	129644790	35357670
17	477638700	129644790
18	1767263190	477638700
19	6564120420	1767263190
20	24466267020	6564120420
21	91482563640	24466267020
22	343059613650	91482563640
23	1289904147324	343059613650
24	48619464041452	1289904147324
25	18367353072152	48619464041452

Note that in the second term, which is 3, it is formed from twice the previous term plus one; in the third term, which is 7, it is formed from twice the previous term plus one; in the fourth term, which is 15, it is formed from twice the previous term plus one, and so on. Therefore, we can construct a recursive formula as follow:

$$M_p = 2M_{p-1} + 1, \ p \geq 1, \qquad M_0 = 0 \tag{11}$$

From Equation (10), for $p = 0$, $M_0 = 2^0 - 1 = 0$, then $M_0 = 0$.

$$\begin{aligned} M_p &= 2^p - 1. \text{Write } M_p \text{ as } M_p = 2 \cdot 2^{p-1} - 1 \\ M_{p-1} &= 2^{p-1} - 1 \\ 2^{p-1} &= M_{p-1} + 1 \\ M_p &= 2 \cdot 2^{p-1} - 1 \\ &= 2(M_{p-1} + 1) - 1 \\ &= 2M_{p-1} + 1 \end{aligned} \tag{12}$$

Thus

$$M_p = 2M_{p-1} + 1$$

- For $p = 1$, then $n = 1$, so that $\begin{bmatrix} 2 \\ 1 \end{bmatrix} = (2 \cdot 1 + 1)(C_1)^2 = 1$
- For $p = 2$, then $n = 3$, so that $\begin{bmatrix} 6 \\ 3 \end{bmatrix} = (2 \cdot 3 + 1)(C_3)^2 = 175$
- For $p = 3$, then $n = 7$, so that $\begin{bmatrix} 14 \\ 7 \end{bmatrix} = (2 \cdot 7 + 1)(C_7)^2 = 2760615$
- For $p = 4$, then $n = 15$, so that $\begin{bmatrix} 30 \\ 15 \end{bmatrix} = (2 \cdot 15 + 1)(C_{15})^2 = 2913690606794775$

Table 5 displays the relationships of Tribinomial coefficients with Triangular and Mersenne Numbers. All numbers on the table are Tribinomial coefficients. The Number in red blocks are Mersenne Numbers. Note that on Table 5, the Mersenne Numbers only appears 3 times, on column 1, 3, and 7 (or in row 2, 6, and 14), because the 4$^{\text{th}}$ Mersenne Number will appear at column 15 or row 30 if we continue constructing the table. The numbers in green blocks are Triangular Numbers. From Table 5 we see that the Triangular numbers appear on the second and second last term of Tribinomial coefficients. Thus, $T_n = \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ n-1 \end{bmatrix}$.

This results can be derive using the straigtforward definition of Tribinomial coefficients for $r = 1$ and $r = n - 1$ as follows:

Table 5. The Tribinomial Coefficients, Triangular Numbers, and Mersenne Numbers

n \ r	0	1	2	3	4	5	6	7	8	9
0	1									
1	1	1								
2	1	3	1							
3	1	6	6	1						
4	1	10	20	10	1					
5	1	15	50	50	15	1				
6	1	21	105	175	105	21	1			
7	1	28	196	490	490	196	28	1		
8	1	36	336	1176	1764	1176	336	36	1	
9	1	45	540	2520	5292	2520	540	45	1	
10	1	55	825	4950	13860	19404	13860	4950	825	55
11	1	66	1210	9075	32670	60984	60984	32670	9075	1210
12	1	78	1716	15730	70785	169884	226512	169884	70785	15730
13	1	91	2366	26026	143143	429429	736164	736164	429429	143143
14	1	105	3185	41405	273273	1002001	2147145	2760615	2147145	1002001
15	1	120	4200	63700	496860	2186184	5725720	9202050	9202050	5725720
16	1	136	5440	95200	866320	4504864	14158144	34763300	27810640	27810640
17	1	153	6936	138720	1456560	8836464	32821152	77364144	118195220	118195220
18	1	171	8721	197676	2372112	16604784	71954064	200443464	367479684	449141836
19	1	190	10830	276165	3755844	30046752	150233760	488259720	1057896060	1551580888
20	1	210	13300	379050	5799465	52581816	300467520	1126753200	2848181700	4936848280
21	1	231	16170	512050	8756055	89311761	578399976	2478857040	7229999700	146206666060
22	1	253	19481	681835	12954865	147685461	1075994073	5226256926	17420856420	40648664980
23	1	276	23276	896126	18818646	238369516	1941008916	10606227291	40067969766	106847919376
24	1	300	27600	1163800	26883780	376372920	3405278800	20796524100	88385227425	267119798440
25	1	325	32500	1495000	37823500	582481900	5824819000	39525557500	187746398125	638337753625

For $r = 1$,

$$\begin{bmatrix} n \\ 1 \end{bmatrix} = \frac{1}{n-1+1}\binom{n+1}{1+1}\binom{n}{1}$$
$$= \frac{1}{n}\binom{n+1}{2}\binom{n}{1} = \frac{1}{n}\frac{(n+1)(n)(n-1)!}{2!(n-1)!}\frac{n(n-1)!}{1!(n-1)!}$$
$$= \frac{n(n+1)}{2} = T_n$$

For $r = n-1$,

$$\begin{bmatrix} n \\ n-1 \end{bmatrix} = \frac{1}{n-(n-1)+1}\binom{n+1}{(n-1)+1}\binom{n}{n-1}$$
$$= \frac{1}{2}\binom{n+1}{n}\binom{n}{n-1} = \frac{1}{2}\frac{(n+1)n!}{1!n!}\frac{n(n-1)!}{1!(n-1)!}$$
$$= \frac{n(n+1)}{2} = T_n$$

4. CONCLUSIONS

Based on the Results and Discussion we can conclude that the Triangular Numbers t_n appears on every second term and second last term of Tribinomial coefficients $\begin{bmatrix} n \\ r \end{bmatrix}$, i.e. as $T_n = \begin{bmatrix} n \\ 1 \end{bmatrix} = \begin{bmatrix} n \\ n-1 \end{bmatrix}$. The Catalan Number appears as the sum of all Tribinomial coefficients $\begin{bmatrix} n \\ r \end{bmatrix}$, for $0 \leq r \leq n$, or $\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} = C_{n+1}$, and Mersenne Numbers appears on the odd CTC.

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