

An Investigation on the Edge-gracefulness of Bubble Sort, Bubble Sort Star, and Kulli-Cycle Windmill Graphs

John Rafael M. Antalan¹, Richard P. Tagle¹, Aaron DC. Angel^{1*}, John Loureynz F. Gamurot^{1,2}

¹Department of Mathematics and Physics, College of Science, Central Luzon State University, Science City of Muñoz, Nueva Ecija, 3120, Philippines

²Department of Mathematics and Computer Science, University of the Philippines Baguio, Baguio City, 2600, Philippines

*Corresponding author: aaron.angel@clsu2.edu.ph

Abstract

A fundamental area of graph theory, graph labeling concerns the assignment of integers to the vertices, edges, or both of a graph under specified constraints. One of the many types of graph labeling is edge-graceful labeling. Edge-graceful labeling requires assigning distinct integers from 1 to q to the edges of a graph G with p vertices and q edges so that the induced vertex labels, obtained by summing the labels of all incident edges modulo p , are pairwise distinct. Now, determining which graph families admit such labeling remains challenging, and this study focuses on the edge-gracefulness of bubble sort graphs, bubble sort star graphs, and Kulli-cycle windmill graphs, whose orders and sizes depend on multiple variables or factorial terms, unlike simpler graphs determined by a single parameter. Using the contrapositive of Lo's Theorem and number-theoretic techniques, we establish that the bubble sort graph B_n is not edge-graceful whenever the parameter n is not an even integer that is greater than or equal to 4, that the only bubble sort star graphs that have the potential to be edge-graceful are BS_n with $n \geq 4$, and that W_3 is the only edge-graceful Kulli-cycle windmill graph.

Keywords

Diophantine, Divisibility, Edge-Gracefulness, Order, Size

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1. INTRODUCTION

Graph theory originated from Leonhard Euler's seminal solution to the Seven Bridges of Königsberg problem in 1736, which became a cornerstone of modern graph theory by representing land regions as vertices and bridges as edges. A graph is generally defined as a mathematical structure consisting of a set of vertices together with a set of edges joining selected pairs of vertices. Formally, a graph G is an ordered pair $G = (V, E)$, where V denotes a nonempty finite set of vertices and E is a finite multiset of unordered pairs of vertices representing the edges (Biggs et al., 1986).

One important research area in graph theory is graph labeling, introduced during the 1960s, which concerns assigning integers to the vertices, edges, or both under prescribed conditions (Gallian, 2022). Over the years, the subject has expanded considerably, with hundreds of labeling variations and thousands of published studies (Ramachandran and Gnanaseelan, 2022). Among the earliest and most widely studied labeling schemes is graceful labeling, where the vertices of a graph with q edges receive distinct integer labels from 0 to q , such that each edge receives the absolute difference of the labels of its

endpoints. A graph is graceful whenever the resulting edge labels are precisely the integers from 1 to q (Venkatesan and Sekar, 2017).

Edge-graceful labeling, introduced by Lo (Lo, 1985), reverses this perspective by assigning labels to the edges first, and vertex labels are determined by the incident edge labels. More precisely, let G be a graph that has p vertices and q edges. The graph is edge-graceful if one can assign the integers 1 through q bijectively to its edges so that the induced vertex labels, obtained by summing the labels of all incident edges modulo p , are pairwise distinct (Venkatesan and Sekar, 2017).

Since Lo introduced edge-graceful labeling in 1985, it has been studied across numerous graph families. Early work by Kuang et al. (1988) focused on unicyclic graphs, while Lee et al. (1988) examined complete graphs. This line of research continued with Lee (1989), who investigated trees, and later with Lee and Seah (1990a); Lee et al. (1990), who considered cycle graphs, their k -th powers, and regular complete k -partite graphs. Building on these results, Lee and Seah (1990b) extended the study to the k -th powers of general graphs. These developments eventually led Lee et al. (1992) to explore the edge-gracefulness of 2-regular graphs formed by the union of

multiple cycles.

Beyond specific graph families, researchers have also examined how edge-graceful labeling behaves under graph operations. An early contribution in this area was made by Lee and Seah (1991), who studied compositions involving step graphs and null graphs. Their work later inspired Shiu et al. (2002) to investigate similar compositions involving paths and null graphs (Shiu et al., 2002). In addition, several variations of edge-graceful labeling have been proposed and widely explored, such as edge-odd graceful labeling (Solairaju and Chithra, 2009), modular edge-graceful labeling (Jones et al., 2012), and edge-even graceful labeling (Elsonbaty and Daoud, 2017).

Recent contributions to the literature include investigations of the edge-gracefulness of wheel graphs (Angel et al., 2024), water wheel-type graphs and their splitting graphs (Angel et al., 2025), and usual fan graphs together with their degree-splitted variants (Antalan et al., 2026).

All of these recent works investigated the edge-gracefulness of specific graph families by using the graphs' size and order, the concept of divisibility, quadratic Diophantine equations, and Lo's Theorem, which was introduced by Sheng-Ping Lo (1985). Lo's Theorem provides a fundamental necessary condition for edge-gracefulness. Its contrapositive serves as an effective sieve for eliminating graphs that cannot be edge-graceful, thereby narrowing the search to graphs that may admit an edge-graceful labeling (Lo, 1985; Tamang, 2021).

Furthermore, most existing studies focus on simple graphs whose order and size are determined by a single parameter. Comparatively little attention has been given to graphs whose order and size are more complex. Despite recent advances in edge-graceful labeling, no study has yet investigated the edge-gracefulness of bubble-sort, bubble-sort star, and Kulli-cycle windmill graphs, whose order and size are governed by multiple parameters or factorial expressions.

Therefore, this paper investigates the edge-gracefulness of bubble-sort, bubble-sort star, and Kulli-cycle windmill graphs using the graphs' order and size, number-theoretic techniques, and the contrapositive of Lo's Theorem. The results offer new insights into these graph families and demonstrate the effectiveness of number-theoretic methods in addressing graph labeling problems.

2. PRELIMINARIES

For completeness, this subsection presents the essential preliminary concepts and results used in this study. Readers who are unfamiliar with the graph-theoretic and number-theoretic concepts employed may consult the standard references by Gallian (2022), Gross et al. (2018), and Burton (2010).

Throughout this paper, let $G = (V, E)$ denote a graph, where V represents its vertex set and E represents its edge set. Graph G has order $p = |V|$ and size $q = |E|$. Unless explicitly stated otherwise, every graph considered in this study is finite, simple, connected, and undirected. Standard notation and terminology from graph theory are adopted throughout (Gallian, 2022; Gross et al., 2018).

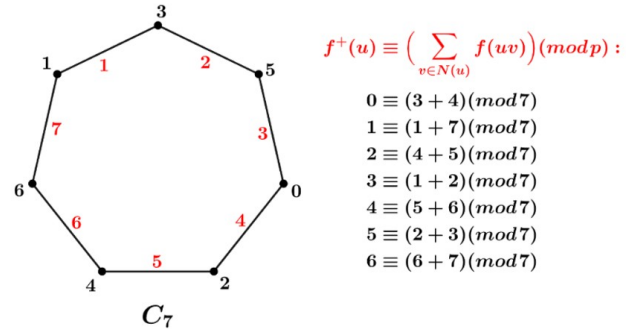


Figure 1. An Edge-Gracefully Labeled Cycle Graph C_7

Definition 3.1 For positive integers a and b with $a \leq b$, we write $a \mid b$ whenever $b = ak$ for some positive integer k . In this case, a is called a divisor of b . If b cannot be expressed in this form, we write $a \nmid b$ and say that a is not a divisor of b .

Definition 3.2 Modular Congruence: Let $m \in \mathbb{Z}^+$, and let $a, b \in \mathbb{Z}$. We say that a is congruent to b modulo m , written as $a \equiv b \pmod{m}$, if $m \mid (a-b)$. Equivalently, we have $a = b + km$ for some $k \in \mathbb{Z}$.

Theorem 3.1 Addition Property of Modular Congruence: If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$ for integers a, b, c, d , then $a + c \equiv b + d \pmod{m}$.

Definition 3.3 Let $G = (V, E)$ be a (p, q) -graph. The graph G is said to be edge-graceful if there exists a bijective function $f : E \rightarrow \{1, 2, \dots, q\}$, such that the induced vertex labeling $f^+ : V \rightarrow \mathbb{Z}_p$, defined by

$$f^+(u) = \left(\sum_{v \in N(u)} f(uv) \right) \pmod{p}$$

is likewise bijective. Here, $\mathbb{Z}_p = \{0, 1, 2, \dots, p-1\}$, and $f(uv)$ corresponds to the label of the edge uv incident with the vertex u in G .

An edge-graceful labeling of cycle graph C_7 is illustrated in Figure 1. Since the graph has seven vertices and seven edges, the edges receive distinct labels from 1 to 7, which produces the distinct vertex labels from 0 to 6 modulo 7, illustrating an edge-graceful labeling.

Definition 3.4 (Wang et al., 2015; Zhang and Lin, 2019) Let B_n , where $n \geq 2$, denote the bubble sort graph. Its vertices correspond to all $n!$ permutations of the set $[n] = \{1, 2, \dots, n\}$. Each vertex is written as $x = x_1x_2 \dots x_n$, where $x_i \in [n]$ and $x_i \neq x_j$ whenever $i \neq j$, for $i, j \in [n]$. Two vertices $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_n$ of B_n are connected precisely when y is obtained from x by swapping two adjacent entries, that is, $y = x_1 \dots x_{i-1}x_ix_{i+1}x_{i+2} \dots x_n$, for some $i \in \{1, 2, \dots, n-1\}$, where y denotes the permutation formed from x by swapping the entries in positions i and $i+1$. Clearly, B_n is $(n-1)$ -regular. Consequently, $|V(B_n)| = n!$ and $|E(B_n)| = \frac{n!(n-1)}{2}$.

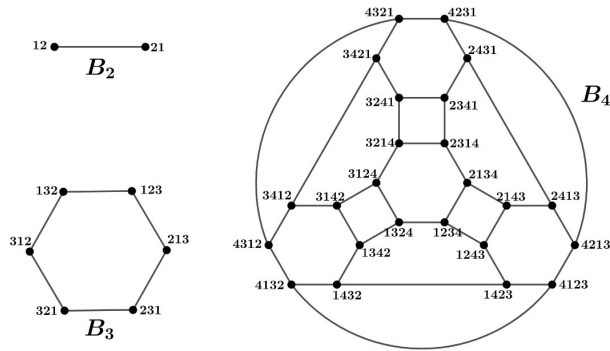


Figure 2. Bubble Sort Graphs B_2 , B_3 , and B_4

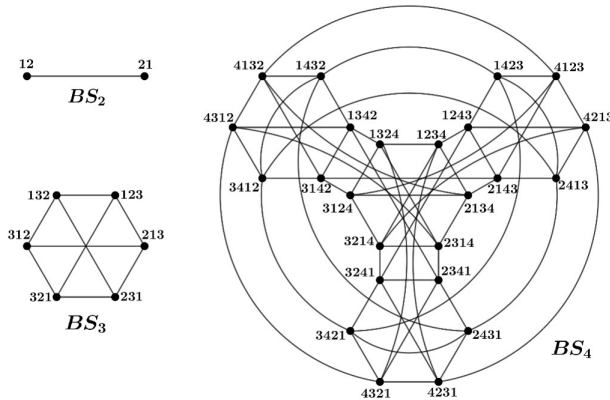


Figure 3. Bubble Sort Star Graphs BS_2 , BS_3 , and BS_4

The graphs B_2 , B_3 , and B_4 are depicted in Figure 2.

As illustrated in Figure 2, graphs B_2 , B_3 , and B_4 are constructed by taking all permutations of the sets $\{1,2\}$, $\{1,2,3\}$, and $\{1,2,3,4\}$, respectively, as their vertex sets. The adjacency between vertices is then determined according to the main conditions specified in Definition 3.4.

Definition 3.5(Guo and Lu, 2016) Let BS_n denote the bubble-sort star graph. Its vertices correspond to all $n!$ permutations of the set $[n] = \{1, 2, \dots, n\}$. Each vertex is written as $x = x_1x_2 \dots x_n$, where $x_i \in [n]$ and $x_i \neq x_j$ whenever $i \neq j$, for $i, j \in [n]$. Two vertices $x = x_1x_2 \dots x_n$ and $y = y_1y_2 \dots y_n$ of BS_n are connected precisely when $y = x \circ (1, i)$, $i \in \{2, 3, \dots, n\}$, or $y = x \circ (i-1, i)$, $i \in \{3, 4, \dots, n\}$, where $x \circ (i, j)$ denotes the permutation obtained from x by swapping the entries in positions i and j . Clearly, BS_n is $(2n-3)$ -regular. Consequently, $|V(BS_n)| = n!$ and $|E(BS_n)| = \frac{n!(2n-3)}{2}$.

The graphs BS_2 , BS_3 , and BS_4 are depicted in Figure 3.

As illustrated in Figure 3, graphs BS_2 , BS_3 , and BS_4 are constructed by taking all permutations of the sets $\{1,2\}$, $\{1,2,3\}$, and $\{1,2,3,4\}$, respectively, as their vertex sets. The adjacency between vertices is then determined according to the main conditions specified in Definition 3.5.

Definition 3.6 (Kulli et al., 2016; Abdullah, 2023) Let C_n be the cycle graph on $n \geq 3$ vertices and let K_1 be the trivial graph

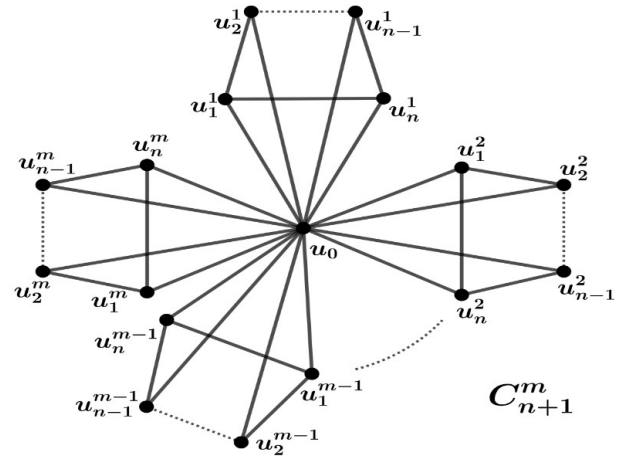


Figure 4. Kulli-Cycle Windmill Graph C_{n+1}^m

consisting of a single vertex. The Kulli-cycle windmill graph, denoted by C_{n+1}^m , is constructed from m copies of the join $C_n + K_1$ and identifying the vertex corresponding to K_1 in each copy as a single common vertex. Consequently, $|V(C_{n+1}^m)| = mn + 1$ and $|E(C_{n+1}^m)| = 2mn$.

Figure 4 presents an illustration of a general Kulli-cycle windmill graph.

Figure 4 illustrates that C_{n+1}^m is constructed by identifying the K_1 vertex from each of the m copies of $C_n + K_1$ into a single common vertex u_0 .

Theorem 3.2 Lo's Theorem (Lo, 1985) If a graph G of p vertices and q edges is edge-graceful, then

$$p \left| \left(q^2 + q - \frac{p(p-1)}{2} \right) \right| \quad (1)$$

A complete proof of Lo's Theorem was provided by (Antalan et al., 2026). By applying the contrapositive of this theorem, one can eliminate graphs with order p and size q that cannot be edge-graceful, thereby narrowing the search to those that may satisfy the property.

Lemma 3.1 (Tamang, 2021) Consider the quadratic Diophantine equation.

$$ax^2 + bxy + cy^2 + dx + ey + f = 0, \quad (2)$$

with integer coefficients a, b, c, d, e , and f , and unknown variables x and y . This equation can be transformed into the equivalent form

$$X^2 - DY^2 = N, \quad (3)$$

where $X = Dy + E$, $Y = 2ax + by + d$, and $N = E^2 - DF$.

Under this change of variables, $D = b^2 - 4ac$, $E = bd - 2ae$, and $F = d^2 - 4af$.

Furthermore, if (X, Y) is a solution to Equation 3, then there exist integers x and y , such that

$$x = \frac{Y - by - d}{2a} \quad \text{and} \quad y = \frac{X - E}{D}. \quad (4)$$

where x and y values in Equation 4 are the integer solutions for Equation 2.

For completeness and to improve readability, the detailed derivation of the transformation from Equation 2 to Equation 3 is presented below.

Derivation: Starting from the general quadratic Diophantine equation

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

First, we group the terms containing x . So, we'll have

$$ax^2 + (by + d)x + (cy^2 + ey + f) = 0.$$

By multiplying both sides by $4a$, we'll have

$$4a^2x^2 + 4a(by + d)x + 4a(cy^2 + ey + f) = 0,$$

which also means that

$$4a^2x^2 + 4ax(by + d) = -4a(cy^2 + ey + f). \quad (5)$$

Now, observe that

$$(2ax + by + d)^2 = 4a^2x^2 + 4ax(by + d) + (by + d)^2. \quad (6)$$

By substituting Equation 5 for Equation 6, we get the equation

$$(2ax + by + d)^2 = -4a(cy^2 + ey + f) + (by + d)^2. \quad (7)$$

Expanding and rearranging Equation 7, we get

$$(2ax + by + d)^2 = -4acy^2 - 4aey - 4af + b^2y^2 + 2bdy + d^2.$$

$$(2ax + by + d)^2 = (b^2 - 4ac)y^2 + (2bd - 4ae)y + (d^2 - 4af). \quad (8)$$

Now, we introduce the following substitutions:

$$D = b^2 - 4ac, \quad E = bd - 2ae, \quad F = d^2 - 4af.$$

By applying the substitutions above, Equation 8 becomes

$$(2ax + by + d)^2 = Dy^2 + 2Ey + F. \quad (9)$$

Multiplying both sides of Equation 9 by D , we obtain

$$D(2ax + by + d)^2 = D^2y^2 + 2DEy + DF.$$

Adding $E^2 - DF$ on both sides of the equation gives

$$D(2ax + by + d)^2 + E^2 - DF = D^2y^2 + 2DEy + E^2. \quad (10)$$

Now, since

$$(Dy + E)^2 = D^2y^2 + 2DEy + E^2,$$

then Equation 10 becomes

$$D(2ax + by + d)^2 + (E^2 - DF) = (Dy + E)^2. \quad (11)$$

Now, we introduce the following substitutions:

$$X = Dy + E, \quad Y = 2ax + by + d, \quad N = E^2 - DF.$$

By applying the substitutions above, Equation 11 becomes

$$DY^2 + N = X^2. \quad (12)$$

Finally, by rearranging Equation 12, we get

$$X^2 - DY^2 = N,$$

which is the desired result.

Remark 3.1 If $c = 0$ in Equation 2, then Equation 3 simplifies to

$$X^2 - (bY)^2 = N. \quad (13)$$

Rewriting Equation 13 in factored form gives

$$(X + bY)(X - bY) = N.$$

Suppose that (N_1, N_2) is a factor pair of N . Then the corresponding linear system is

$$(X + bY) = N_1 \quad \text{and} \quad (X - bY) = N_2.$$

Solving this system for X and Y yields

$$X = \frac{N_1 + N_2}{2} \quad \text{and} \quad Y = \frac{N_1 - N_2}{2b}.$$

3. RESULT AND DISCUSSION

The principal results of this paper are discussed in this section, beginning with a result concerning the edge-gracefulness of bubble sort graphs.

Theorem 4.1 Let B_n be a bubble sort graph on $n!$ vertices. If n is not an even integer that is greater than or equal to 4, then B_n is not edge-graceful.

Proof By Definition 3.4, we have $p = n!$ and $q = \frac{n!(n-1)}{2}$. Thus, setting $p = n!$ and $q = \frac{n!(n-1)}{2}$ in Equation 1, of Theorem 3.2 we obtain

$$\begin{aligned}
 n! & \left| \left[\left(\frac{n!(n-1)}{2} \right)^2 + \left(\frac{n!(n-1)}{2} \right) - \frac{n!((n!)-1)}{2} \right] \right. \\
 n! & \left| \left[\frac{(n!)^2(n-1)^2}{4} + \frac{n!(n-1) - n!((n!)-1)}{2} \right] \right. \\
 n! & \left| \left\{ n! \left[\frac{n!(n-1)^2}{4} + \frac{(n-1) - ((n!)-1)}{2} \right] \right\} \right. \\
 n! & \left| \left\{ n! \left[\frac{n!(n-1)^2}{4} + \frac{n-n!}{2} \right] \right\}, \right.
 \end{aligned}$$

which further implies that

$$\frac{n! [n!(n-1)^2 + 2(n-n!)]}{4(n!)} \in \mathbb{Z}$$

$$\frac{n!(n^2 - 2n + 1) + 2n - 2(n!)}{4} \in \mathbb{Z}$$

$$\frac{n!(n^2 - 2n + 1) - 2(n!) + 2n}{4} \in \mathbb{Z}$$

$$\frac{n!(n^2 - 2n - 1) + 2n}{4} \in \mathbb{Z}.$$

Now, we proceed by considering two cases.

Case 1: Let $n \geq 4$.

Since $n!$ contains the factor 4 for all $n \geq 4$, it follows that

$$n!(n^2 - 2n - 1) \equiv 0 \pmod{4}.$$

It follows from Theorem 3.1 that

$$n!(n^2 - 2n - 1) + 2n \equiv 2n \pmod{4}.$$

Thus,

$$n!(n^2 - 2n - 1) + 2n \equiv 0 \pmod{4}$$

if and only if

$$2n \equiv 0 \pmod{4},$$

which is equivalent to n being even.

Hence, for $n \geq 4$, we have

$$n!(n^2 - 2n - 1) + 2n \equiv 0 \pmod{4}$$

if and only if n is even.

Case 2: Let $n \in \{1, 2, 3\}$.

We evaluate each case:

$$\begin{aligned}
 n = 1 : \quad & n!(n^2 - 2n - 1) + 2n \\
 & = 1!(1^2 - 2(1) - 1) + 2(1) = 0 \pmod{4}, \\
 n = 2 : \quad & n!(n^2 - 2n - 1) + 2n \\
 & = 2!(2^2 - 2(2) - 1) + 2(2) = 2 \pmod{4}, \\
 n = 3 : \quad & n!(n^2 - 2n - 1) + 2n \\
 & = 3!(3^2 - 2(3) - 1) + 2(3) = 18 \equiv 2 \pmod{4}.
 \end{aligned}$$

Hence, for $n \in \{1, 2, 3\}$, we have $n!(n^2 - 2n - 1) + 2n \equiv 0 \pmod{4}$ if and only if $n = 1$.

Thus, for $n \geq 1$, we have $n!(n^2 - 2n - 1) + 2n \equiv 0 \pmod{4}$ if and only if $n = 1$ or n is an even integer greater than or equal to 4. However, B_1 is a trivial graph, so it cannot be edge-graceful. Therefore, we conclude that by the contrapositive of the Lo's Theorem, if n is not an even integer greater than or equal to 4, then B_n is not edge-graceful.

Theorem 4.1 further implies that the only bubble sort graphs with potential to be edge-graceful are B_n , where n is an even integer with $n \geq 4$.

Theorem 4.2. Let BS_n be a bubble sort star graph on $n!$ vertices. If $n < 4$, then BS_n is not edge-graceful.

Proof. By Definition 3.5, we have $|V(BS_n)| = n!$ and $|E(BS_n)| = \frac{n!(2n-3)}{2}$. Thus, setting $p = n!$ and $q = \frac{n!(2n-3)}{2}$ in Equation (1) of Theorem 3.2, we obtain

$$\begin{aligned}
 n! & \left| \left[\left(\frac{n!(2n-3)}{2} \right)^2 + \left(\frac{n!(2n-3)}{2} \right) - \frac{n!((n!)-1)}{2} \right] \right. \\
 n! & \left| \left[\frac{(n!)^2(2n-3)^2}{4} + \frac{n!(2n-3) - n!((n!)-1)}{2} \right] \right. \\
 n! & \left| \left\{ n! \left[\frac{n!(2n-3)^2}{4} + \frac{(2n-3) - ((n!)-1)}{2} \right] \right\} \right. \\
 n! & \left| \left\{ n! \left[\frac{n!(2n-3)^2 + 4n - 4 - 2n!}{4} \right] \right\}, \right.
 \end{aligned}$$

which further implies that

$$\frac{n! [n!(2n-3)^2 + 4(n-1) - 2(n!)]}{4(n!)} \in \mathbb{Z}$$

$$\frac{n!(4n^2 - 12n + 9) + 4(n-1) - 2(n!)}{4} \in \mathbb{Z}$$

$$\frac{n!(4n^2 - 12n + 9) - 2(n!) + 4(n-1)}{4} \in \mathbb{Z}$$

$$\frac{n!(4n^2 - 12n + 7) + 4(n-1)}{4} \in \mathbb{Z}.$$

Consequently, we seek all positive integers n for which

$$n!(4n^2 - 12n + 7) + 4(n-1) \equiv 0 \pmod{4}.$$

Now, we proceed by considering two cases.

Case 1: Let $n \geq 4$.

Since $n!$ contains the factor 4 for all $n \geq 4$, it follows that

$$n!(4n^2 - 12n + 7) \equiv 0 \pmod{4}.$$

Furthermore,

$$4(n-1) \equiv 0 \pmod{4}.$$

Consequently,

$$n!(4n^2 - 12n + 7) + 4(n-1) \equiv 0 \pmod{4}$$

for all $n \geq 4$.

Case 2: Let $n \in \{1, 2, 3\}$.

We evaluate each case: **Case 2:** Let $n \in \{1, 2, 3\}$.

We evaluate each case:

$$\begin{aligned} n = 1 : \quad & n!(4n^2 - 12n + 7) + 4(n-1) \\ &= 1! \left(4(1)^2 - 12(1) + 7 \right) + 4(1-1) \\ &= -1 \equiv 3 \pmod{4}, \end{aligned}$$

$$\begin{aligned} n = 2 : \quad & n!(4n^2 - 12n + 7) + 4(n-1) \\ &= 2! \left(4(2)^2 - 12(2) + 7 \right) + 4(2-1) \\ &= 2 \equiv 2 \pmod{4}, \end{aligned}$$

$$\begin{aligned} n = 3 : \quad & n!(4n^2 - 12n + 7) + 4(n-1) \\ &= 3! \left(4(3)^2 - 12(3) + 7 \right) + 4(3-1) \\ &= 50 \equiv 2 \pmod{4}. \end{aligned}$$

Hence,

$$n!(4n^2 - 12n + 7) + 4(n-1) \not\equiv 0 \pmod{4}$$

Thus, for $n \geq 1$, we have

$$n!(4n^2 - 12n + 7) + 4(n-1) \equiv 0 \pmod{4}$$

if and only if $n \geq 4$.

Therefore, we conclude that by the contrapositive of Lo's Theorem, if $n < 4$, then BS_n is not edge-graceful.

Theorem 4.2 establishes that graphs BS_1 , BS_2 , and BS_3 are not edge-graceful. Moreover, it further implies that the only

bubble sort star graphs with potential to be edge-graceful are BS_n , where $n \geq 4$.

Theorem 4.3 Among all Kulli-cycle windmill graphs C_{n+1}^m , only C_4^1 is edge-graceful.

Proof By Definition 3.6, we have $|V(C_{n+1}^m)| = mn + 1$ and $|E(C_{n+1}^m)| = 2mn$. Thus, setting $p = mn + 1$ and $q = 2mn$ in Equation 1 of Theorem 3.2, we obtain

$$\begin{aligned} (mn+1) & \left| \left[(2mn)^2 + (2mn) - \frac{(mn+1)(mn)}{2} \right] \right| \\ (mn+1) & \left| \left[4m^2n^2 + (2mn) - \frac{(mn+1)(mn)}{2} \right] \right| \\ (mn+1) & \left| \left[\frac{8m^2n^2 + 4mn - m^2n^2 - mn}{2} \right] \right| \\ (mn+1) & \left| \left[\frac{7m^2n^2 + 3mn}{2} \right] \right|. \end{aligned}$$

Consequently, we obtain

$$\frac{7m^2n^2 + 3mn}{2(mn+1)} \in \mathbb{Z}$$

$$\frac{7m^2n^2 + 3mn}{2mn+2} \in \mathbb{Z}$$

Letting $mn = t$, the expression simplifies to

$$\frac{7t^2 + 3t}{2t + 2} \in \mathbb{Z}$$

This means

$$7t^2 + 3t = (2t + 2)k, \quad \text{for some } k \in \mathbb{Z}.$$

Therefore, we seek all integer solutions (t, k) to the Diophantine equation

$$7t^2 + 3t - 2tk - 2k = 0. \quad (14)$$

Observe that Equation 14 is a particular case of Equation 2, with $a = 7$, $b = -2$, $c = 0$, $d = 3$, $e = -2$, and $f = 0$.

By applying Lemma 3.1 and Remark 3.1, we get

$$D = 4, \quad E = 22, \quad F = 9, \quad N = 448.$$

Consequently, Equation 14 simplifies to

$$X^2 - 4Y^2 = 448,$$

where

$$X = 4k + 22 \quad \text{and} \quad Y = 14t - 2k + 3.$$

Therefore,

$$X = \frac{N_1 + N_2}{2}, \quad Y = \frac{N_1 - N_2}{-4},$$

$$k = \frac{X - 22}{4}, \quad t = \frac{Y + 2k - 3}{14},$$

where (N_1, N_2) are the factor pairs of $N = 448$. First, all possible factor pairs (N_1, N_2) of $N = 448$ are exhaustively considered. Now, using the four equations above, corresponding values of X and Y are computed for each factor pair, after which the associated values of k and t are obtained. The resulting (t, k) pairs constitute all candidate integer solutions for Equation 14. The complete set of integer solutions for Equation 14 is presented in Table 1.

Table 1 shows that the complete set of integer solutions for $7t^2 + 3t - 2tk - 2k = 0$ is

$$(t, k) \in \{(3, 9), (0, 0), (-5, -20), (-2, -11)\},$$

which are the cells colored in yellow. Recall that $t = mn$, where m and n are both positive integers. Hence, $t > 0$. Among the four solutions, only $(3, 9)$, where $t = 3$, satisfies this condition, making it the only admissible solution.

Thus, by the contrapositive of Lo's Theorem, if $mn \neq 3$, then C_{n+1}^m is not edge-graceful. It follows that C_4^1 is the only candidate that may be edge-graceful. Now, observe that

$$C_4^1 \cong C_3 + K_1 \cong W_3,$$

where W_3 denotes the wheel graph having three outer vertices. Since W_3 is edge-graceful (Angel et al., 2024), it follows that C_4^1 is also edge-graceful. An explicit edge-graceful labeling of C_4^1 is provided in Figure 6.

```
#Python Program for C_4^1~(1)
from itertools import permutations
numbers = [1, 2, 3, 4, 5, 6]
def compute_v_values(u0u1, u0u2, u0u3, u1u2, u2u3, u3u1):
    u0 = (u0u1 + u0u2 + u0u3) % 4
    u1 = (u1u2 + u0u1 + u3u1) % 4
    u2 = (u1u2 + u2u3 + u0u2) % 4
    u3 = (u12u3 + u13u1 + u0u3) % 4
    return u0, u1, u2, u3
def backtrack(perm, used):
    if len(perm) == 6:
        v_values = compute_v_values(*perm)
        if len(set(v_values)) == 4:
            print(f"Permutation: {perm}, u0, u1, u2 and u3: {v_values}")
            return
    for num in numbers:
        if not used[num - 1]:
            used[num - 1] = True
            perm.append(num)
            backtrack(perm, used)
            perm.pop()
            used[num - 1] = False
def find_valid_permutations():
    used = [False] * 6
    backtrack([], used)
find_valid_permutations()
```

Figure 5. Python Algorithm for Identifying a Valid Edge-Graceful Labeling Of C_4^1

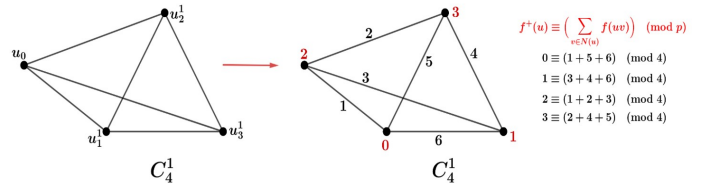


Figure 6. An edge-gracefully labeled C_4^1

As illustrated in Figure 6, C_4^1 has four vertices and six edges. The edges were labeled with distinct integers from 1 to 6, producing distinct induced vertex labels from 0 to 4, satisfying the conditions of an edge-graceful labeling.

The labeling illustrated in Figure 6 is one of the outputs generated by the custom-developed Python program in Figure 5. The Python program in Figure 5 implements a backtracking algorithm that systematically explores all possible assignments of the integers 1 to 6 to the edges of C_4^1 in search of a valid edge-graceful labeling. Therefore, C_4^1 is the only edge-graceful Kulli-cycle windmill graph. These findings provide opportunities for further investigation of edge-gracefulness within the graph families examined. Since this investigation was limited to the application of the contrapositive of Lo's Theorem in identifying non-edge-graceful members of the graph families considered, future research may focus on completing the characterization of the remaining candidate graphs, exploring related graph families and graph-labeling variants, and employing alternative approaches to edge-gracefulness.

4. CONCLUSIONS

Lo's Theorem provides a fundamental necessary condition for edge-gracefulness, and its contrapositive served as the basis for this investigation. Using this approach, the study established that a necessary condition for the edge-gracefulness of bubble sort graphs B_n is that n be an even integer greater than or equal to 4, that a necessary condition for the edge-gracefulness of bubble sort star graphs BS_n is that $n \geq 4$, and that C_4^1 is the only edge-graceful Kulli-cycle windmill graph. These findings extend the current understanding of edge-graceful labeling by establishing new necessary conditions for the edge-gracefulness of bubble sort and bubble sort star graphs and providing a complete characterization of the edge-gracefulness of Kulli-cycle windmill graphs, thereby contributing to the broader development of graph labeling theory.

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Table 1. Solutions for $7t^2 + 3t - 2tk - 2k = 0$

N_1	N_2	X	Y	t	k	N_1	N_2	X	Y	t	k
1	448	224.5	111.75	15	50.6	-1	-448	-224.5	-111.8	-17	-61.63
2	224	113	55.5	7	22.8	-2	-224	-113	-55.5	-9	-33.8
4	112	58	27	3	9	-4	-112	-58	-27	-5	-20
7	64	35.5	14.25	1.3	3.4	-7	-64	-35.5	-14.3	-3.3	-14.4
8	56	32	12	1	2.5	-8	-56	-32	-12	-3	-13.5
14	32	23	4.5	0.1	0.3	-14	-32	-23	-4.5	-2	-11
16	28	22	3	0	0	-16	-28	-22	-3	-2	-11
448	1	224.5	-111.8	-15	-50.6	-448	-1	-224.5	111.8	17	61.6
224	2	113	-55.5	-7	22.8	-224	-2	-113	55.5	1.07	-33.8
112	4	58	-27	-4	9	-112	-4	-58	27	-1.14	-20
64	7	35.5	-14.25	-0.8	3.4	-64	-7	-35.5	14.3	-1.25	-14.4
56	8	32	-12	-0.7	2.5	-56	-8	-32	12	-1.3	-13.5
32	14	23	-4.5	-0.5	0.3	-32	-14	-23	4.5	-1.5	-11.3
28	16	22	-3	-0.4	0	-28	-16	-22	3	-1.6	-11

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