

# Jacobson-Type Graphs Over Rings: Induced Subgraphs and Isomorphisms

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## Abstract

Jacobson-type graphs provide a combinatorial framework for studying structural properties of finite commutative rings. We address two foundational questions: the behavior of these graphs under subring inclusions, and conditions under which distinct Jacobson-type constructions are isomorphic. This paper proves that whenever  $S$  is a subring of a finite commutative ring  $R$ , the Jacobson-type graphs of  $S$  arise naturally as induced subgraphs of the corresponding graphs of  $R$ , with finiteness playing an essential role in sustaining this embedding. We further characterize precisely when the Jacobson graph, the  $n$ -array Jacobson graph, and the matrix Jacobson graph over a ring are isomorphic. These results establish a correspondence between algebraic relations of rings and combinatorial relations of their associated graphs.

## Keywords

Jacobson Graph, Jacobson Radical, Induced Subgraph, Graph Isomorphism, Matrix Rings, Finite Commutative Rings

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## 1. INTRODUCTION

Graphs built from algebraic structures create a bridge between ring theory and combinatorics, allowing purely algebraic properties to be viewed through a geometric lens. The broader tradition of associating graphs with commutative rings dates back to Beck (1988), which first introduced a graph on a commutative ring, using zero divisors to define adjacency, primarily in the context of graph coloring. Anderson and Naseer (1993) later studied Beck's coloring in greater depth, before Anderson and Livingston (1999) proposed the now-standard formulation of the zero-divisor graph, which has since become the main reference point for the field. Parallel efforts to encode ring structure via graphs were also pursued by Sharma and Bhatwadekar (1995), who defined adjacency based on the non unit elements of a ring, an approach that directly anticipated the construction later formalized as the Jacobson graph. The Jacobson graph, introduced by Azimi et al. (2013), builds on this tradition by replacing the zero-divisor condition with one governed by the Jacobson radical, thereby capturing finer ring-theoretic information through its adjacency structure. This construction has since attracted sustained attention, with researchers investigating various graph parameters. These include diameter and matching (Ghayour et al., 2018), girth and clique number (Akbari et al., 2015), planarity (Parsapour et al., 2015), as well as vertex degrees (Fattahi et al., 2019), isomorphisms (Azimi et al., 2014), and Hamiltonian properties (Azimi and Farrokhi

D. G., 2017).

Motivated by this framework, several extensions of the Jacobson graph have emerged. Matrix Jacobson graphs,  $n$ -array Jacobson graphs, and their higher-dimensional counterparts preserve the original idea of translating algebraic relations into adjacency conditions, while adding new structural properties that arise from matrix rank, dimension, and determinantal constraints (Ghayour et al., 2017; Humaira et al., 2020, 2022, 2023). Parallel developments in the study of graphs associated with matrix rings have also been explored, notably the zero-divisor graphs of  $M_n(R)$  examined in (Akbari and Mohammadian, 2004) and the planarity results discussed in (Akbari et al., 2003). Furthermore, Anderson et al. (2011) demonstrated that passing from a ring to its matrix ring introduces qualitatively new graph-theoretic properties, particularly concerning vertex degrees stratified by matrix rank. These observations motivate a closer examination of how Jacobson-type constructions behave under analogous algebraic operations, such as subring inclusions and ring-theoretic decompositions. Precise definitions of each construction used in this paper are collected in Section 2 for reference.

Despite the growing body of work on these extensions, two foundational questions have received little systematic attention. The first concerns the subring-subgraph correspondence: given an inclusion  $S \subseteq R$  of finite commutative rings, does the Jacobson-type graph associated with  $S$  embed naturally

as a subgraph of the one associated with  $R$ , and if so, in what sense? The second concerns isomorphism: when do distinct Jacobson-type graph constructions over the same ring, or over different rings, produce isomorphic graphs? Addressing these questions in a unified manner is the central aim of this paper.

In Section 3.1, we show that whenever  $S$  is a subring of a finite commutative ring  $R$ , the graphs  $\mathfrak{J}_S$ ,  $\mathfrak{J}_{M_n(S)}$ ,  $\mathfrak{J}_S^n$ , and  $\mathfrak{J}_S^{n \times n}$  naturally appear as induced subgraphs of the corresponding graphs over  $R$ . We identify the precise role that finiteness plays in preserving this embedding. Section 3.2 then provides a complete characterization of the conditions under which the various Jacobson-type graph constructions over a ring become isomorphic. Together, these results yield a coherent account of the isomorphism classes and induced embedding properties of Jacobson-type constructions.

## 2. EXPERIMENTAL SECTION

Throughout this paper, all rings are assumed to be finite and commutative with identity, unless explicitly stated otherwise. The finiteness assumption is not merely conventional; it plays a decisive role in several of the main results, as discussed in Remark 3.2. We use standard ring-theoretic terminology from (McDonald, 1974; Herstein, 1991), and graph-theoretic terminology from (Diestel, 2017). In particular, all graphs considered are simple and undirected. For background on subgraph relations and graph isomorphism problems, we refer the reader to Diestel (2017).

Let  $R$  be a commutative ring with identity  $I_R$ , and let  $U(R)$  denote the group of units of  $R$ . The Jacobson radical of  $R$ , denoted  $J(R)$ , is the intersection of all maximal ideals of  $R$ .

**Definition 2.1.** (Azimi et al., 2013) The Jacobson graph of  $R$ , denoted by  $\mathfrak{J}_R$ , is a graph whose vertex set is  $R \setminus J(R)$ , and two distinct vertices  $x, y$  are adjacent whenever  $I_R - xy \notin U(R)$ .

This construction fits naturally within the broader framework of ring-theoretic graphs initiated by Beck (1988) and Anderson and Livingston (1999), in which algebraic properties of a ring are encoded as adjacency conditions on a graph built from its elements. The defining feature of the Jacobson graph is its use of the Jacobson radical  $J(R)$  rather than the set of zero divisors, as the structural parameter governing both the vertex set and the adjacency rule. Elements of  $J(R)$  are excluded from the vertex set since, by the definition of the Jacobson radical, for any  $x \in J(R)$  and  $y \in R$ , the element  $I_R - xy$  is a unit in  $R$ . Thus, the condition  $I_R - xy \notin U(R)$  is never satisfied, meaning any element in  $J(R)$  would be an isolated vertex (adjacent to no other vertices) and therefore combinatorially trivial. In this work, we always write a vector in  $R^n$  as a column vector.

**Definition 2.2.** (Ghayour et al., 2017) The  $n$ -array Jacobson graph of  $R$ , denoted by  $\mathfrak{J}_R^n$ , is a graph whose vertex set is  $R^n \setminus J(R)^n$  (viewed as column vectors), and two distinct vertices  $\mathbf{x}, \mathbf{y}$  are adjacent whenever  $1 - \mathbf{x}^T \mathbf{y} \notin U(R)$ .

**Definition 2.3.** (Humaira et al., 2022) The matrix Jacobson graph of  $R$ , denoted by  $\mathfrak{J}_R^{n \times n}$ , is a graph whose vertex set is  $M_n(R) \setminus M_n(J(R))$  and two distinct matrices  $A, B$  are adjacent whenever  $I_R - \det(AB) \notin U(R)$ .

The matrix Jacobson graph  $\mathfrak{J}_R^{n \times n}$  may be regarded as a Jacobson-type analog of the zero-divisor graph of the matrix ring  $M_n(R)$  studied by Akbari and Mohammadian (2004). While both constructions use non-invertibility to define adjacency, the Jacobson-type variant employs the determinant as its algebraic invariant, introducing a richer stratification of the vertex set according to matrix rank, as studied systematically in (Humaira et al., 2020, 2022).

**Definition 2.4.** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs. The graphs  $G_1$  and  $G_2$  are isomorphic, denoted by  $G_1 \cong G_2$ , if there exists a bijective function  $f : V_1 \rightarrow V_2$  such that  $(u, v) \in E_1$  if and only if  $(f(u), f(v)) \in E_2$  for all  $u, v \in V_1$ .

This definition ensures that adjacency and non-adjacency are preserved, implying that the two graphs are structurally identical despite differences in labeling. A natural extension of this notion is obtained by restricting the correspondence to a portion of the vertices and edges of a larger graph. This leads to the concept of subgraph isomorphism, which plays a central role in structural graph theory. The specific notion of an induced subgraph, in which adjacency is fully inherited from the ambient graph, is the primary subgraph relation used in this paper, as it preserves both edges and non-edges between the selected vertices.

**Definition 2.5.** Let  $G_1 = (V_1, E_1)$  and  $G_2 = (V_2, E_2)$  be two graphs. The graph  $G_1$  is subgraph-isomorphic to  $G_2$  if there exists a bijection  $f : V_1 \rightarrow V' \subseteq V_2$  such that if  $(u, v) \in E_1$ , then  $(f(u), f(v)) \in E_2$ . If  $|V_1| \neq |V_2|$  or  $|E_1| \neq |E_2|$ , we say  $G_1$  is proper subgraph-isomorphic to  $G_2$ .

Several variations of subgraph isomorphism are commonly studied:

- The graph  $G_1$  is *spanning subgraph-isomorphic* to  $G_2$  if  $G_1$  is subgraph-isomorphic to  $G_2$  and  $|V_1| = |V_2|$ .
- The graph  $G_1$  is a  *$V'$  induced subgraph-isomorphic* to  $G_2$  if  $f : V_1 \rightarrow V' \subseteq V_2$  is a bijection satisfying  $(u, v) \in E_1$  if and only if  $(f(u), f(v)) \in E_2$ . Thus,  $G_1$  corresponds exactly to an induced subgraph of  $G_2$ .

The set of all  $n \times n$  upper and lower triangular matrices over  $R$  is denoted by  $UT_n(R)$  and  $LT_n(R)$ , respectively. The set of diagonal matrices is denoted by  $D_n(R)$  and is defined in Equation (1):

$$D_n(R) = \{A = \text{diag}(x_1, x_2, \dots, x_n) \mid x_i \in R\}. \quad (1)$$

The notation defined in Equation (1) will be used throughout the paper.

## 3. RESULT AND DISCUSSION

### 3.1 Subring and Subgraph of Jacobson-type Graphs

A careful analysis of how a subring interacts with the Jacobson-type graph of a ring requires comparing units, non units, and

Jacobson radicals of both rings. Since adjacency is determined by the (non-)invertibility of certain expressions, these comparisons become essential. We begin with some basic lemmas.

**Lemma 3.1.** *Let  $R$  be a finite ring with identity  $I_R$ , and let  $S \subseteq R$  be a subring. Then  $S \setminus U(S) \subseteq R \setminus U(R)$ .*

*Proof.* In a finite commutative ring, an element is a unit if and only if it is not a zero divisor. Let  $x \in S \setminus U(S)$ . Then  $x$  is a zero divisor in  $S$ , so there exists a non zero  $y \in S$  such that  $xy = 0$ . Since  $S \subseteq R$ , the same equation holds in  $R$ . Hence,  $x$  is a zero divisor in  $R$ , so  $x \notin U(R)$ .  $\square$

**Remark 3.2.** *The finiteness assumption in Lemma 3.1 is essential. If  $R$  is infinite, a non unit of  $S$  may become a unit of  $R$ . For example, take  $S = \mathbb{Z} \subseteq \mathbb{Q} = R$ . Then  $\mathbb{Q} \setminus U(\mathbb{Q}) = \{0\}$ ,  $\mathbb{Z} \setminus U(\mathbb{Z}) = \mathbb{Z} \setminus \{-1, 1\}$ . Thus  $2 \in S \setminus U(S)$ , but  $2 \in U(R)$ .*

**Lemma 3.3.** *Let  $R$  be a finite ring with identity  $I_R$ , and let  $S$  be a subring of  $R$  with identity  $I_S$ . The following statements are satisfied.*

- (i) *If  $I_S = I_R$ , then  $S \cap U(R) = U(S)$ .*
- (ii) *If  $I_S \neq I_R$ , then  $S \cap U(R) = \emptyset$ .*

*Proof.*

- (i)  $I_S = I_R$ . It is clear that  $U(S) \subseteq S \cap U(R)$ . Conversely, let  $u \in S \cap U(R)$ . Then there exists  $v \in R$  such that  $uv = I_R$ . Without loss of generality, consider the left-multiplication map  $\ell_u : S \rightarrow S$  defined by  $s \mapsto us$ . If  $\ell_u(s) = 0$ , multiplying by  $v$  gives  $s = v(us) = 0$ . Thus,  $\ell_u$  is injective. Since  $S$  is finite,  $\ell_u$  is also surjective; hence, there exists  $s_0 \in S$  such that  $us_0 = I_S$ . Thus,  $u$  is invertible in  $S$ .
- (ii) Suppose  $I_S \neq I_R$ . Assume, for contradiction, that there exists  $u \in S \cap U(R)$ . Let  $v = u^{-1} \in R$ . Then  $I_S = I_S I_R = I_S(uv) = (I_S u)v = I_R$ , contradicting  $I_S \neq I_R$ . Therefore,  $S \cap U(R) = \emptyset$ .  $\square$

**Lemma 3.4.** *Let  $R$  be a finite local ring with identity  $I_R$  and a unique maximal ideal  $\mathfrak{m}_R$ . If  $S \subseteq R$  is a commutative subring containing an identity  $I_S$ , then*

- (i)  $I_S = I_R$ ,
- (ii)  $S$  is local, and
- (iii) the unique maximal ideal  $\mathfrak{m}_S$  of  $S$  satisfies  $\mathfrak{m}_S = S \cap \mathfrak{m}_R$ .

*Proof.* (i) Since  $I_S$  is the multiplicative identity of  $S \subseteq R$ , then  $I_S$  is an idempotent of  $R$ . Since  $R$  is finite, it is Artinian (see Atiyah and Macdonald (1969), Proposition 8.2). Moreover, in a commutative Artinian local ring, the only idempotents are 0 and  $I_R$  (see Atiyah and Macdonald (1969), Exercise 8.2; see also Matsumura (1986), Theorem 2.7). Since  $I_S$  is a non zero idempotent, it follows that  $I_S = I_R$ .

- (ii) Since any idempotent of  $S$  is also an idempotent of  $R$ ,  $S$  has no non trivial idempotents. For a finite commutative Artinian ring, the absence of non trivial idempotents is equivalent to indecomposability. Moreover, an Artinian

commutative ring is local if and only if it is indecomposable (see Atiyah and Macdonald (1969), Proposition 8.7 and Corollary 8.8; see also Matsumura (1986), Theorem 2.7). Therefore,  $S$  is local.

- (iii) Since  $R$  is a finite commutative ring and  $S$  is a subring of  $R$  with the same identity, we have  $U(S) = S \cap U(R)$  by Lemma 3.3 (i). Therefore,  $\mathfrak{m}_S = S \setminus U(S) = S \setminus (S \cap U(R)) = S \cap (R \setminus U(R)) = S \cap \mathfrak{m}_R$ .  $\square$

In the next lemma, we give a useful relationship between the Jacobson radicals of a ring and its subring.

**Lemma 3.5.** *Let  $R$  be a finite ring and  $S$  be a subring of  $R$ . Then  $S \setminus J(S) \subseteq R \setminus J(R)$ .*

*Proof.* We show  $S \cap J(R) \subseteq J(S)$ . By Proposition 1.9 in (Atiyah and Macdonald, 1969), an element  $x \in J(R)$  if and only if  $I_R - yx$  is a unit in  $R$  for all  $y \in R$ . Let  $y \in S$ . Since  $S \subseteq R$ ,  $y$  is also an element of  $R$ . Since  $x \in J(R)$  and  $y \in R$ , the element  $I_R - yx$  must be a unit in  $R$ . Next, we consider two cases as follows.

*Case 1.*  $I_S = I_R$ . If  $I_R - yx$  is a unit in  $R$ , then it cannot be a non unit in  $S$ , by Lemma 3.1. Thus,  $I_S - yx$  must be a unit in  $S$ .

*Case 2.*  $I_S \neq I_R$ . Since  $S \cap U(R) = \emptyset$  by Lemma 3.3 (ii), then  $I_R - yx \notin S$ . Since  $S$  is a subring,  $I_S - yx \in S$ , then  $I_S - yx \notin U(R)$ . We now show that  $I_S - yx \in U(S)$ . Assume to the contrary that  $I_S - yx \notin U(S)$ . Then there exists a non zero  $z_0 \in S$  with  $z_0 \notin U(R)$  such that  $(I_S - yx)z_0 = 0$ . However,  $(I_R - yx)z_0 = z_0 - yxz_0 = I_S z_0 - yxz_0 = (I_S - yx)z_0 = 0$ . Since  $z_0$  is non zero, then  $I_R - yx \notin U(R)$ , which is a contradiction.

Since  $I_S - yx$  is a unit in  $S$  for all  $y \in S$ , it follows that  $x \in J(S)$ . This implies  $S \cap J(R) \subseteq J(S)$ . The statement  $S \setminus J(S) \subseteq R \setminus J(R)$  is equivalent to  $S \cap J(R) \subseteq J(S)$ , which completes the proof.  $\square$

**Theorem 3.6.** *Let  $R$  be a finite ring and  $S$  be a subring of  $R$  with identity  $I_S$ . Then*

1.  $\mathfrak{J}_S$  is an  $(S \setminus J(S))$ -induced subgraph of  $\mathfrak{J}_R$ ,
2.  $\mathfrak{J}_S^n$  is an  $(S^n \setminus J(S)^n)$ -induced subgraph of  $\mathfrak{J}_R^n$ , and
3.  $\mathfrak{J}_S^{n \times n}$  is an  $(M_n(S) \setminus M_n(J(S)))$ -induced subgraph of  $\mathfrak{J}_R^{n \times n}$ .

*Proof.* By Lemma 3.5,  $V(\mathfrak{J}_S) = S \setminus J(S) \subseteq R \setminus J(R) = V(\mathfrak{J}_R)$ . Let  $x, y \in V(\mathfrak{J}_S)$ . Suppose  $x$  and  $y$  are adjacent in  $\mathfrak{J}_S$ . Then  $I_S - xy \notin U(S)$ . If  $I_S = I_R$ , then  $I_R - xy = I_S - xy$ , and by Lemma 3.1, it is also a non unit in  $R$ . If  $I_S \neq I_R$ , the same argument as in Lemma 3.5 shows that  $I_R - xy \notin U(R)$ . Thus,  $x$  and  $y$  are adjacent in  $\mathfrak{J}_R$ .

Suppose  $x, y \in V(\mathfrak{J}_S)$  and  $I_S - xy \in U(S)$ . If  $I_R - xy \notin U(R)$ , then (as in the proof of Lemma 3.4) we reach a contradiction because this would imply  $I_S - xy$  is also a non unit of  $S$ . Therefore,  $I_R - xy \in U(R)$ , so  $x$  and  $y$  are not adjacent in  $\mathfrak{J}_R$ . Thus,  $\mathfrak{J}_S$  is induced in  $\mathfrak{J}_R$ . The array and matrix cases follow by the same argument, with the dot product and determinant replacing multiplication, respectively.  $\square$

By Theorem 3.6, we derive a necessary and sufficient condition for a ring and its subring to share the same Jacobson graphs. Consequently, by Remark 3.2, Theorem 3.6 does not hold in general for infinite rings.

**Corollary 3.7.** *Let  $S$  be a distinct subring unity of a ring  $R$ . The graph  $\mathfrak{J}_S = \mathfrak{J}_R$  if and only if  $S \setminus J(S) = R \setminus J(R)$ .*

*Proof.* If  $\mathfrak{J}_S = \mathfrak{J}_R$ , then  $S \setminus J(S) = V(\mathfrak{J}_S) = V(\mathfrak{J}_R) = R \setminus J(R)$ . Conversely, if  $R \setminus J(R) = S \setminus J(S)$ , then  $V(\mathfrak{J}_R) = V(\mathfrak{J}_S)$ . Since  $\mathfrak{J}_S$  is an induced subgraph of  $\mathfrak{J}_R$ , then  $\mathfrak{J}_R = \mathfrak{J}_S$ .  $\square$

For a ring  $R$ , the set of all  $n \times n$  invertible matrices with entries in  $R$ , together with matrix multiplication as the group operation, is denoted by  $GL_n(R)$ . In other words,  $GL_n(R) = \{A \in M_n(R) \mid A \text{ is invertible over } R\}$ .

Let  $(R, \mathfrak{m})$  be a local commutative ring with unique maximal ideal  $\mathfrak{m}$ . It is clear that  $J(R) = \mathfrak{m}$ . We denote  $\bar{R} = R/J(R)$  as the quotient field of  $R$  with respect to  $J(R)$ , where  $\bar{R} = \{\bar{a} = a + J(R) \mid a \in R\}$  with  $\bar{a} = a + J(R) = \{a + x \mid x \in J(R)\}$ . Since  $\bar{R}$  is a field, then  $U(\bar{R}) = \bar{R} \setminus \{\bar{0}\}$ .

Let  $u(\bar{R}) = \{\bar{a} \in U(\bar{R}) \mid \bar{a}^{-1} = \bar{a}\}$  and  $u'(\bar{R}) = \{\bar{a} \in U(\bar{R}) \mid \bar{a}^{-1} \neq \bar{a}\}$ . The vertex set of square matrix Jacobson graph,  $\mathfrak{J}_R^{n \times n}$ , can be partitioned as shown in Equation (2):

$$V(\mathfrak{J}_R^{n \times n}) = V_{\bar{0}} \cup V_{u(\bar{R})} \cup V_{u'(\bar{R})}, \quad (2)$$

where  $V_{\bar{0}} = \{A \in V(\mathfrak{J}_R^{n \times n}) \mid \overline{\det(A)} = \bar{0}\}$ ,  $V_{u(\bar{R})} = \{A \in V(\mathfrak{J}_R^{n \times n}) \mid \overline{\det(A)} \in u(\bar{R})\}$ , and  $V_{u'(\bar{R})} = \{A \in V(\mathfrak{J}_R^{n \times n}) \mid \overline{\det(A)} \in u'(\bar{R})\}$ .

**Theorem 3.8.** (Humaira et al., 2022) *Let  $(R, \mathfrak{m})$  be a local ring. The graph  $\mathfrak{J}_R^{n \times n}$  is a disjoint union of the following components:*

- (i)  $|V_{\bar{0}}|$  isolated vertices,
- (ii)  $|u(\bar{R})|$  complete subgraphs, where the vertex set  $V_{\bar{x}}$  forms a complete subgraph for each  $\bar{x} \in u(\bar{R})$ ,
- (iii)  $\frac{|u'(\bar{R})|}{2}$  complete bipartite subgraphs, where the vertex set  $V_{\bar{x}} \cup V_{\bar{x}^{-1}}$  forms a complete bipartite subgraph for each  $\bar{x} \in u'(\bar{R})$ .

If a graph  $G$  has isolated vertices with vertex set  $V_{\bar{0}}$ , we define  $G^*$  to be the graph obtained from  $G$  by removing all isolated vertices, such that the vertex set of  $G^*$  is  $V(G^*) = V(G) \setminus V_{\bar{0}}$ .

**Theorem 3.9.** *Let  $(R, \mathfrak{m})$  be a local ring. Then  $\mathfrak{J}_{GL_n(R)}$  is a spanning subgraph of  $(\mathfrak{J}_R^{n \times n})^*$  if and only if  $R/J(R) \cong \mathbb{F}_2$ .*

*Proof.* Let  $R/J(R) \cong \mathbb{F}_2$ . Then by Theorem 3.8 and Equation (2), the graph  $\mathfrak{J}_R^{n \times n}$  consists of isolated vertices with vertex set  $V_{\bar{0}} = \{A \in V(\mathfrak{J}_R^{n \times n}) \mid \overline{\det(A)} = \bar{0}\}$  and a complete subgraph  $(\mathfrak{J}_R^{n \times n})^*$  with vertex set  $V_1 = \{A \in V(\mathfrak{J}_R^{n \times n}) \mid \overline{\det(A)} = \bar{1}\}$ . Therefore,  $V(\mathfrak{J}_{GL_n(R)}) = V((\mathfrak{J}_R^{n \times n})^*) = GL_n(R)$ , and it is clear that  $\mathfrak{J}_{GL_n(R)}$  is a spanning subgraph of  $(\mathfrak{J}_R^{n \times n})^*$ .

Assume that  $|R/J(R)| \geq 3$ . Then there exist non zero  $x, y \in R$  such that  $\bar{x} \neq \bar{y}$  and  $\bar{x}, \bar{y} \in R/J(R)$ . By Theorem 3.8,  $\mathfrak{J}_R^{n \times n}$  has at least one isolated vertex set  $V_{\bar{0}}$  and two complete

components with vertex sets  $V_{\bar{x}}$  and  $V_{\bar{y}}$ , where  $V_{\bar{x}} = \{A \in V(\mathfrak{J}_R^{n \times n}) \mid \det(A) \in x + \mathfrak{m}\}$  and  $V_{\bar{y}} = \{A \in V(\mathfrak{J}_R^{n \times n}) \mid \det(A) \in y + \mathfrak{m}\}$ . Consider the matrices  $A = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix} \in V_{\bar{x}}$  and  $B = \begin{pmatrix} 1 & 0 \\ 0 & y \end{pmatrix} \in V_{\bar{y}}$ . Then  $AB \in E(\mathfrak{J}_{GL_n(R)})$ , but  $AB \notin E(\mathfrak{J}_R^{n \times n})$ . So,  $\mathfrak{J}_{GL_n(R)}$  is not a spanning subgraph of  $(\mathfrak{J}_R^{n \times n})^*$ .  $\square$

### 3.2 Isomorphism in Jacobson-type Graphs of a Ring

Isomorphism plays a central role in graph theory, providing a means to compare graphs based solely on their structural characteristics. Two graphs are isomorphic when they share the same adjacency pattern, differing only by a relabeling of vertices. In this section, we study when different Jacobson-type graphs associated to a ring are isomorphic, and how these isomorphisms relate to the algebraic structure of the underlying rings.

The *disjunctive product* of two graphs  $G$  and  $H$ , denoted by  $G \oplus H$ , is defined on the vertex set  $V(G) \times V(H)$ . Two vertices  $(g, h)$  and  $(g', h')$  are considered adjacent if either  $gg' \in E(G)$  or  $hh' \in E(H)$ . Similar to the Cartesian and direct products of two graphs, this operation is commutative. The concept was first introduced by Ore (1962) under the name *Cartesian sum of graphs*, while the term *disjunctive product* was later adopted in subsequent literature (Scheinerman and Ullman, 1997).

The following proposition establishes a relationship between  $\mathfrak{J}_R$  and  $\mathfrak{J}_{D_n(R)}$ .

**Proposition 3.10.** *Let  $R$  be a finite commutative ring. The  $n$ -disjunctive product of  $\mathfrak{J}_R$ , denoted by  $\mathfrak{J}_R^{\oplus n} = \mathfrak{J}_R \oplus \mathfrak{J}_R \oplus \dots \oplus \mathfrak{J}_R$ , is  $D_n(R \setminus J(R))$  induced subgraph-isomorphic to  $\mathfrak{J}_{D_n(R)}$ . Moreover,  $\mathfrak{J}_R^{\oplus n}$  is  $D_n(R \setminus J(R))$  induced subgraph-isomorphic to  $\mathfrak{J}_{M_n(R)}$ .*

*Proof.* Since  $V(\mathfrak{J}_R) = R \setminus J(R)$ , we have  $V(\mathfrak{J}_R^{\oplus n}) = R \setminus J(R) \times \dots \times R \setminus J(R)$ . Define a map

$$f : V(\mathfrak{J}_R^{\oplus n}) \rightarrow V(\mathfrak{J}_{D_n(R \setminus J(R))}) \subset V(\mathfrak{J}_{D_n(R)}) \quad (3)$$

by  $f(x_1, x_2, \dots, x_n) = \text{diag}(x_1, x_2, \dots, x_n) \in D_n(R \setminus J(R))$ . The map in Equation (3) is a bijection. If  $(x_1, x_2, \dots, x_n)$  and  $(y_1, \dots, y_n)$  are adjacent in  $\mathfrak{J}_R^{\oplus n}$ , then there exists an index  $i$  such that  $x_i$  and  $y_i$  are adjacent in  $\mathfrak{J}_R$ . This implies  $I_R - x_i y_i \notin U(R)$ . Thus, the matrices  $\text{diag}(x_1, x_2, \dots, x_n)$  and  $\text{diag}(y_1, y_2, \dots, y_n)$  are adjacent in  $\mathfrak{J}_{D_n(R)}$ . The converse clearly holds as well, establishing the first isomorphism.

Moreover,  $D_n(R)$  is a subring of  $M_n(R)$  sharing the same identity matrix  $I_n$ . By Lemma 3.3 (i),  $D_n(R) \cap U(M_n(R)) = U(D_n(R))$ . For any two matrices  $A, B \in D_n(R \setminus J(R))$ , the matrix  $I_n - AB$  is diagonal. Thus,  $I_n - AB \notin U(D_n(R))$  if and only if  $I_n - AB \notin U(M_n(R))$ . Consequently, the adjacency relation is strictly preserved when viewed in the larger matrix ring, meaning  $\mathfrak{J}_R^{\oplus n}$  is also  $D_n(R \setminus J(R))$  induced subgraph-isomorphic to  $\mathfrak{J}_{M_n(R)}$ .  $\square$

**Theorem 3.11.** (Azimi et al., 2014) *Let  $R$  and  $R'$  be finite commutative rings. Then  $\mathfrak{J}_R \cong \mathfrak{J}_{R'}$  if and only if  $|J(R)| = |J(R')|$  and  $R/J(R) \cong R'/J(R')$ .*

**Corollary 3.12.** Let  $(R, \mathfrak{m})$  be a local ring. If  $R_1 = R \oplus \cdots \oplus R$  is an  $n$  decomposition of  $R$ , then  $\mathfrak{J}_{R_1}$  is isomorphic to  $\mathfrak{J}_{D_n(R)}$ .

*Proof.* Let  $f$  be a map from  $V(\mathfrak{J}_{R_1})$  to  $V(\mathfrak{J}_{D_n(R)})$  defined by  $f(x_1, x_2, \dots, x_n) = \text{diag}(x_1, x_2, \dots, x_n)$ . Let  $(x_1, x_2, \dots, x_n)$  and  $(y_1, y_2, \dots, y_n)$  be adjacent in  $\mathfrak{J}_{R_1}$ . Then there exists an index  $i$  such that  $I_R - x_i y_i \notin U(R)$ . Thus, the matrix  $\text{diag}(I_R - x_1 y_1, \dots, I_R - x_n y_n) \notin U(D_n(R))$ . It follows that  $\text{diag}(x_1, x_2, \dots, x_n)$  and  $\text{diag}(y_1, y_2, \dots, y_n)$  are adjacent in  $\mathfrak{J}_{D_n(R)}$ .  $\square$

Next, we characterize when the Jacobson graphs of matrix rings over  $R$  and  $R'$  are isomorphic.

**Theorem 3.13.** Let  $R$  and  $R'$  be two finite rings. Then  $\mathfrak{J}_{M_n(R)}$  and  $\mathfrak{J}_{M_n(R')}$  are isomorphic if and only if  $|J(R)| = |J(R')|$  and  $R/J(R) \cong R'/J(R')$ .

*Proof.* Let  $|J(R)| = |J(R')|$ . Then  $M_n(J(R))$  and  $M_n(J(R'))$  have the same cardinality. Since  $R/J(R) \cong R'/J(R')$ , there exists a bijection  $T : R/J(R) \rightarrow R'/J(R')$ . Now, we define  $L : M_n(R/J(R)) \rightarrow M_n(R'/J(R'))$  such that  $L(A) = B$ , where  $A = (a_{ij})$  and  $B = (b_{ij})$  with  $b_{ij} = T(a_{ij})$ . It is clear that  $L$  is a bijection. Therefore,  $M_n(R)/M_n(J(R)) \cong M_n(R/J(R)) \cong M_n(R'/J(R')) \cong M_n(R')/M_n(J(R'))$ . Since  $\mathfrak{J}_{M_n(R)}$  and  $\mathfrak{J}_{M_n(R')}$  are undirected graphs, Theorem 3.11 implies that  $\mathfrak{J}_{M_n(R)} \cong \mathfrak{J}_{M_n(R')}$ .

Conversely, let  $\phi : \mathfrak{J}_{M_n(R)} \rightarrow \mathfrak{J}_{M_n(R')}$  be a graph isomorphism. By Theorem 3.9 in (Humaira et al., 2023), vertices with rank 1 have the minimum degree, and vertices with rank  $n$  have the maximum degree. All vertices at the same rank can have one of two possible degrees. Without loss of generality,  $\phi$  maps the subgraph with vertex set  $\mathbf{e}_1 \mathbf{e}_1^T R$  onto  $\mathbf{e}_1 \mathbf{e}_1^T R'$ . Consequently,  $\mathfrak{J}_R \cong \mathfrak{J}_{R'}$ . By Theorem 3.11, we conclude that  $|J(R)| = |J(R')|$  and  $R/J(R) \cong R'/J(R')$ .  $\square$

**Theorem 3.14.** The graph  $\mathfrak{J}_R^n$  is isomorphic to  $\mathfrak{J}_{D_n(R)}$  if and only if  $R = \mathbb{F}_2$  and  $n = 2$ .

*Proof.* It is obvious that  $\mathfrak{J}_{\mathbb{F}_2}^2$  is isomorphic to  $\mathfrak{J}_{D_2(R)}$ , since both graphs are paths of length 2.

Conversely, assume that  $\mathfrak{J}_R^n \cong \mathfrak{J}_{D_n(R)}$ , but either  $n \neq 2$  or  $R \neq \mathbb{F}_2$ . Let  $\lambda : V(\mathfrak{J}_R^n) \rightarrow V(\mathfrak{J}_{D_n(R)})$  be a graph isomorphism. We consider several cases based on the values of  $n$  and  $|R|$  as follows.

- a. If  $n = 2$  and  $|R| \geq 3$ , then there exists  $x \in R$  such that  $N[(1, 1)^T] = \{(0, 1)^T, (1, 0)^T, (x, y)^T \mid x^{-1} = y \text{ and } x, y \notin \{0, 1\}\}$ . Meanwhile, the neighborhood of  $\text{diag}(1, 1)$  can be expressed as in Equation (4):

$$N[\text{diag}(1, 1)] = \{\text{diag}(1, x), \text{diag}(x, 1) \mid x \in R\}, \quad (4)$$

which implies that  $\deg[(1, 1)^T] < \deg[\text{diag}(1, 1)]$ .

- b. If  $n = 3$  and  $R = \mathbb{F}_2$ , then we obtain  $N[(1, 1, 1)^T] = \{(1, 0, 0)^T, (0, 1, 0)^T, (0, 0, 1)^T\}$ , while the neighborhood set of  $\text{diag}(1, 1, 1)$  is  $N[\text{diag}(1, 1, 1)] = \{\text{diag}(x, y, z) \mid x = 1 \text{ or } y = 1 \text{ or } z = 1, x, y, z \in R\}$ . So, it is clear that  $\deg[(1, 1, 1)^T] = 3 < 6 = \deg[\text{diag}(1, 1, 1)]$ .

- c. If  $n > 3$  and  $R = \mathbb{F}_2$ , then  $\deg[(1, 1, 1, 0, \dots, 0)^T] = 3(2^{n-3}) < 6(2^{n-3}) = \deg[\text{diag}(1, 1, 1, 0, \dots, 0)]$ .
- d. If  $n > 3$  and  $|R| \geq 3$ , then we get  $\deg[(1, 1)^T]|R|^{n-2} < \deg[\text{diag}(1, 1)]|R|^{n-2}$ . Thus,  $\deg[(1, 1, 0, \dots, 0)^T] < \deg[\text{diag}(1, 1, 0, \dots, 0)]$ .

In all cases, we have  $\deg[(1, \dots, 1)^T] < \deg[\text{diag}(1, \dots, 1)]$ , which contradicts the assumption that  $\lambda$  is an isomorphism.  $\square$

More specific details regarding subgraph isomorphisms between the  $n$ -array Jacobson graph and the Jacobson graph of a diagonal matrix are explained below.

**Theorem 3.15.** The graph  $\mathfrak{J}_R^n$  is a proper spanning subgraph-isomorphic to  $\mathfrak{J}_{D_n(R)}$  if and only if  $R = \mathbb{F}_2$  and  $n \geq 3$ .

*Proof.* We construct a map  $T : V(\mathfrak{J}_{\mathbb{F}_2}^n) \rightarrow V(\mathfrak{J}_{D_n(\mathbb{F}_2)})$  by  $T(a_1, a_2, \dots, a_n)^T = \text{diag}(a_1, a_2, \dots, a_n)$ . Since  $T$  is surjective and  $|V(\mathfrak{J}_{\mathbb{F}_2}^n)| = |V(\mathfrak{J}_{D_n(\mathbb{F}_2)})|$ , then  $T$  is a bijection.

If  $(a_1, a_2, \dots, a_n)^T$  is adjacent to  $(b_1, b_2, \dots, b_n)^T$ , then  $\sum_{i=1}^n a_i b_i = 1$ . Therefore, there exists an odd number of indices  $i$  ( $1 \leq i \leq n$ ) such that  $a_i = b_i = 1$ . Hence,  $T(a_1, a_2, \dots, a_n)^T$  is adjacent to  $T(b_1, b_2, \dots, b_n)^T$  in  $\mathfrak{J}_{D_n(\mathbb{F}_2)}$ . Therefore,  $\mathfrak{J}_R^n$  is spanning subgraph-isomorphic to  $\mathfrak{J}_{D_n(R)}$ . The properness follows from the fact that if the number of such indices  $i$  is even, then  $T(a_1, a_2, \dots, a_n)^T$  is adjacent to  $T(b_1, b_2, \dots, b_n)^T$  in  $\mathfrak{J}_{D_n(\mathbb{F}_2)}$ , but  $(a_1, a_2, \dots, a_n)^T$  is not adjacent to  $(b_1, b_2, \dots, b_n)^T$  in  $\mathfrak{J}_{\mathbb{F}_2}^n$ .

Conversely, assume that  $\mathfrak{J}_R^n$  is a proper spanning subgraph-isomorphic to  $\mathfrak{J}_{D_n(R)}$  and either  $R \neq \mathbb{F}_2$  or  $n = 2$ . We consider the following cases:

- a. If  $R = \mathbb{F}_2$  and  $n = 2$ , then by Theorem 3.14,  $\mathfrak{J}_R^n$  is isomorphic to  $\mathfrak{J}_{D_n(R)}$ , contradicting the properness assumption.
- b. If  $R \neq \mathbb{F}_2$  and  $n = 2$ , then there exists an  $x \in R \setminus \{0, 1\}$  such that  $(1, 1)^T$  is adjacent to  $(x, x)^T$ , and  $I_R - x \in U(R)$ . However,  $T(1, 1)^T = \text{diag}(1, 1)$  is not adjacent to  $T(x, x)^T = \text{diag}(x, x)$  in  $\mathfrak{J}_{D_2(R)}$ , which is a contradiction.
- c. If  $R \neq \mathbb{F}_2$  and  $n \geq 3$ , then there exists  $x \in R \setminus \{0, 1\}$  with  $2x = 1$  such that  $(1, 1, 0, \dots, 0)^T$  is adjacent to  $(x, x, 0, \dots, 0)^T$  and  $I_R - x \in U(R)$ . Therefore, we obtain  $T(1, 1, 0, \dots, 0)^T = \text{diag}(1, 1, 0, \dots, 0)$  is not adjacent to  $T(x, x, 0, \dots, 0)^T = \text{diag}(x, x, 0, \dots, 0)$  in  $\mathfrak{J}_{D_n(R)}$ , a contradiction.  $\square$

Previously, we examined the relationship between the Jacobson graph over the matrix ring and the matrix Jacobson graph over the ring, but only in the case of its invertible elements (matrices with rank equal to  $n$ ). We will now examine the adjacency rules in the set of matrices with rank less than  $n$  in  $\mathfrak{J}_{M_k(R)}$ , involving the graph  $\mathfrak{J}_{M_k(R)}$  where  $k < n$ .

Let  $\mathbf{e}_i \in R^n$  be a standard basis vector with the  $i$ -th coordinate being 1 and the others being 0. Let  $S_n$  be the symmetric

group on the set  $\{1, 2, \dots, n\}$ . For  $1 \leq k \leq n$ , we define the subsets of  $M_n(R)$  as in Equation (5) and Equation (6):

$$M_{l_k} = \left\{ \sum_{m=1}^k \mathbf{v}_{i_m} \mathbf{e}_{i_m}^T \mid \mathbf{v}_{i_m} \in R^n \setminus \{\mathbf{0}\}, i_1, \dots, i_k \in S_n \right\} \quad (5)$$

and

$$M_{r_k} = \left\{ \sum_{m=1}^k \mathbf{e}_{i_m} \mathbf{v}_{i_m}^T \mid \mathbf{v}_{i_m} \in R^n \setminus \{\mathbf{0}\}, i_1, \dots, i_k \in S_n \right\}. \quad (6)$$

If  $A \in M_{l_k}$ , then  $A^T \in M_{r_k}$ . For every  $A \in M_n(R)$  with  $\text{rk}(A) = k < n$ ,  $A$  is always equivalent to some  $B \in M_{l_k}$  or  $B \in M_{r_k}$ . In other words, there exist invertible matrices  $P$  and  $Q$  such that  $B = PAQ$ .

**Example 3.16.** Let  $k = 1$ , and let  $\mathbf{v}_1 = (v_{11}, v_{12}, \dots, v_{1n})^T$  and  $\mathbf{v}_2 = (v_{21}, v_{22}, \dots, v_{2n})^T$  be vectors in  $R^n$ . Then, the matrix product  $\mathbf{v}_1 \mathbf{e}_1^T$  is defined in Equation (7):

$$\mathbf{v}_1 \mathbf{e}_1^T = \begin{pmatrix} v_{11} & 0 & \dots & 0 \\ v_{12} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ v_{1n} & 0 & \dots & 0 \end{pmatrix} \in M_{l_1}, \quad (7)$$

and its corresponding transposed form  $\mathbf{e}_2 \mathbf{v}_2^T$  is given by Equation (8):

$$\mathbf{e}_2 \mathbf{v}_2^T = \begin{pmatrix} 0 & 0 & \dots & 0 \\ v_{21} & v_{22} & \dots & v_{2n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = (\mathbf{v}_2 \mathbf{e}_2^T)^T \in M_{r_1}. \quad (8)$$

Furthermore, the multiplications of these matrices can be explicitly evaluated as demonstrated in Equation (9) and Equation (10), respectively:

$$\begin{aligned} (\mathbf{v}_1 \mathbf{e}_1^T)(\mathbf{v}_2 \mathbf{e}_2^T) &= \begin{pmatrix} 0 & v_{11}v_{21} & 0 & \dots & 0 \\ 0 & v_{12}v_{21} & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & v_{1n}v_{21} & 0 & \dots & 0 \end{pmatrix} \\ &= \mathbf{v}_1 v_{21} \mathbf{e}_2^T \in M_{l_1}, \end{aligned} \quad (9)$$

$$\begin{aligned} (\mathbf{e}_1 \mathbf{v}_1^T)(\mathbf{e}_2 \mathbf{v}_2^T) &= \begin{pmatrix} v_{12}v_{21} & v_{12}v_{22} & \dots & v_{12}v_{2n} \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} \\ &= \mathbf{e}_1 v_{12} \mathbf{v}_2^T \in M_{r_1}. \end{aligned} \quad (10)$$

**Lemma 3.17.** Let  $A, B \in V(\mathfrak{J}_{M_n(R)})$ . If  $A$  and  $B$  are adjacent, then  $A^T$  and  $B^T$  are adjacent in  $\mathfrak{J}_{M_n(R)}$ .

*Proof.* Suppose  $I_n - AB \notin U(M_n(R))$ . Since transposition preserves invertibility,  $I_n - B^T A^T = I_n - (AB)^T = (I_n - AB)^T \notin U(M_n(R))$ . Thus,  $A^T$  is adjacent to  $B^T$ .  $\square$

**Lemma 3.18.** For  $1 \leq k < n$ , let  $A$  be the sum of  $\sum_{m=1}^k \mathbf{v}_{i_m} \mathbf{e}_{i_m}^T$  and  $\sum_{m=k+1}^n \mathbf{e}_{i_m} \mathbf{v}_{i_m}^T$ . Then  $A \notin U(M_n(R))$  if and only if

$$\begin{pmatrix} v_{i_1, i_1} & v_{i_2, i_1} & \dots & v_{i_k, i_1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{i_1, i_k} & v_{i_2, i_k} & \dots & v_{i_k, i_k} \end{pmatrix} \notin U(M_k(R)). \quad (11)$$

*Proof.* For  $A = \sum_{m=1}^k \mathbf{v}_{i_m} \mathbf{e}_{i_m}^T + \sum_{m=k+1}^n \mathbf{e}_{i_m} \mathbf{v}_{i_m}^T$ , we apply elementary row and column operations such that  $A$  is equivalent to the block matrix  $B$  in Equation (12):

$$B = \begin{pmatrix} C_1 & \mathbf{0} \\ C_2 & I_{n-k} \end{pmatrix}, \quad (12)$$

where the submatrices  $C_1$  and  $C_2$  are explicitly defined in Equation (13) and Equation (14):

$$C_1 = \begin{pmatrix} v_{i_1, i_1} & v_{i_2, i_1} & \dots & v_{i_k, i_1} \\ \vdots & \vdots & \ddots & \vdots \\ v_{i_1, i_k} & v_{i_2, i_k} & \dots & v_{i_k, i_k} \end{pmatrix} \quad (13)$$

$$C_2 = \begin{pmatrix} v_{i_1, i_{k+1}} & v_{i_2, i_{k+1}} & \dots & v_{i_k, i_{k+1}} \\ \vdots & \vdots & \ddots & \vdots \\ v_{i_1, i_n} & v_{i_2, i_n} & \dots & v_{i_k, i_n} \end{pmatrix}. \quad (14)$$

If  $A \notin U(M_n(R))$ , then  $B \notin U(M_n(R))$  and  $\text{rk}(B) < n$ . This implies  $\text{rk}(C_1) < k$  yielding  $C_1 \notin U(M_k(R))$ , as we get in Equation (11).  $\square$

The following theorem establishes a relationship between  $\mathfrak{J}_{M_n(R)}$  and  $\mathfrak{J}_{M_k(R)}$  for  $1 \leq k < n$ .

**Theorem 3.19.** For a natural number  $n$  and  $1 \leq i, j, k \leq n-1$ , let  $\mathbf{v}_i = (v_{i1}, \dots, v_{in})^T$  and  $\mathbf{u}_j = (u_{j1}, \dots, u_{jn})^T$  be vectors in  $R^n$ . Then the following statements are equivalent.

- (i)  $\sum_{r=1}^k \mathbf{v}_{i_r} \mathbf{e}_{i_r}^T$  and  $\sum_{r=1}^k \mathbf{u}_{j_r} \mathbf{e}_{j_r}^T$  are adjacent in  $\mathfrak{J}_{M_n(R)}$ .
- (ii)  $\sum_{r=1}^k \mathbf{e}_{i_r} \mathbf{v}_{i_r}^T$  and  $\sum_{r=1}^k \mathbf{e}_{j_r} \mathbf{u}_{j_r}^T$  are adjacent in  $\mathfrak{J}_{M_n(R)}$ .
- (iii) The matrices

$$\begin{pmatrix} v_{i_1, j_1} & \dots & v_{i_k, j_1} \\ v_{i_1, j_2} & \dots & v_{i_k, j_2} \\ \vdots & \ddots & \vdots \\ v_{i_1, j_k} & \dots & v_{i_k, j_k} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} u_{j_1, i_1} & \dots & u_{j_k, i_1} \\ u_{j_1, i_2} & \dots & u_{j_k, i_2} \\ \vdots & \ddots & \vdots \\ u_{j_1, i_k} & \dots & u_{j_k, i_k} \end{pmatrix}$$

are adjacent in  $\mathfrak{J}_{M_k(R)}$ .

*Proof.* According to Lemma 3.17, statement (i) holds if and only if statement (ii) is satisfied.

Let  $\sum_{r=1}^k \mathbf{v}_{ir} \mathbf{e}_{ir}^T$  and  $\sum_{r=1}^k \mathbf{u}_{jr} \mathbf{e}_{jr}^T$  be adjacent in  $\mathfrak{J}_{M_n(R)}$ . Using the multiplication approach demonstrated in Examples 3.16 and Equation (9), we obtained Equation (15):

$$\left( \sum_{r=1}^k \mathbf{v}_{ir} \mathbf{e}_{ir}^T \right) \left( \sum_{p=1}^k \mathbf{u}_{jp} \mathbf{e}_{jp}^T \right) = \sum_{p=1}^k \left( \sum_{r=1}^k \mathbf{v}_{ir} u_{jp,ir} \right) \mathbf{e}_{jp}^T. \quad (15)$$

Consequently, the expansion of the identity matrix minus this product can be evaluated through Equation (16)

$$\begin{aligned} I_n - \left( \sum_{r=1}^k \mathbf{v}_{ir} \mathbf{e}_{ir}^T \right) \left( \sum_{p=1}^k \mathbf{u}_{jp} \mathbf{e}_{jp}^T \right) &= \sum_{q=1}^n \mathbf{e}_q \mathbf{e}_q^T - \sum_{p=1}^k \left( \sum_{r=1}^k \mathbf{v}_{ir} u_{jp,ir} \right) \mathbf{e}_{jp}^T \\ &= \sum_{p=1}^k \left( \mathbf{e}_{jp} - \sum_{r=1}^k \mathbf{v}_{ir} u_{jp,ir} \right) \mathbf{e}_{jp}^T + \sum_{p=k+1}^n \mathbf{e}_{jp} \mathbf{e}_{jp}^T. \end{aligned} \quad (16)$$

By Lemma 3.18, we have Equation (17)

$$I_k - \begin{pmatrix} v_{i_1, j_1} & \cdots & v_{i_k, j_1} \\ v_{i_1, j_2} & \cdots & v_{i_k, j_2} \\ \vdots & \ddots & \vdots \\ v_{i_1, j_k} & \cdots & v_{i_k, j_k} \end{pmatrix} \begin{pmatrix} u_{j_1, i_1} & \cdots & u_{j_k, i_1} \\ u_{j_1, i_2} & \cdots & u_{j_k, i_2} \\ \vdots & \ddots & \vdots \\ u_{j_1, i_k} & \cdots & u_{j_k, i_k} \end{pmatrix} \notin U(M_k(R)). \quad (17)$$

Thus, statement (iii) is proved. Conversely, statement (iii) holds if and only if statement (i) also applies.  $\square$

**Example 3.20.** Consider two vertices  $\begin{pmatrix} 0 & 1 \\ 2 & 2 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$  which are adjacent in  $\mathfrak{J}_{M_2(\mathbb{Z}_3)}$ . By Theorem 3.19, we can determine that the following pairs of matrices constitute the edges in  $\mathfrak{J}_{M_4(\mathbb{Z}_3)}$ :

$$\begin{aligned} 1. & \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ 2. & \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \text{ and} \\ 3. & \begin{pmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 2 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \end{aligned}$$

By applying Theorem 3.19 with  $k = 1$ , we obtain the following corollary.

**Corollary 3.21.** For a natural number  $n$  and  $1 \leq i, j \leq n$ , let  $\mathbf{v}_i = (v_{i1}, \dots, v_{in})^T$ ,  $\mathbf{u}_j = (u_{j1}, \dots, u_{jn})^T \in R^n$ . Then the following statements are equivalent:

1.  $\mathbf{v}_i \mathbf{e}_i^T$  and  $\mathbf{u}_j \mathbf{e}_j^T$  are adjacent in  $\mathfrak{J}_{M_n(R)}$ .

2.  $\mathbf{e}_i \mathbf{v}_i^T$  and  $\mathbf{e}_j \mathbf{u}_j^T$  are adjacent in  $\mathfrak{J}_{M_n(R)}$ .
3.  $v_{ij}$  and  $u_{ji}$  are adjacent in  $\mathfrak{J}_R$ .

Furthermore, we have the following corollary, which describes the relation of two adjacent vertices in  $\mathfrak{J}_R^n$  with two adjacent vertices in  $\mathfrak{J}_{M_n(R)}$ .

**Corollary 3.22.** Let  $\mathbf{v}, \mathbf{u} \in R^n$ . The vertices  $\mathbf{u}$  and  $\mathbf{v}$  are adjacent in  $\mathfrak{J}_R^n$  if and only if, for any  $\mathbf{x}_m \in R^n$ , at least one of the following pairs of vertices is adjacent in  $\mathfrak{J}_{M_n(R)}$ .

1.  $\mathbf{u} \mathbf{e}_i^T$  and  $\mathbf{e}_i \mathbf{v}^T + \sum_{m=1, m \neq i}^n \mathbf{e}_m \mathbf{x}_m^T$ ;
2.  $\mathbf{v} \mathbf{e}_i^T$  and  $\mathbf{e}_i \mathbf{u}^T + \sum_{m=1, m \neq i}^n \mathbf{e}_m \mathbf{x}_m^T$ ;
3.  $\mathbf{e}_i \mathbf{u}^T$  and  $\mathbf{v} \mathbf{e}_i^T + \sum_{m=1, m \neq i}^n \mathbf{x}_m \mathbf{e}_m^T$ ;
4.  $\mathbf{e}_i \mathbf{v}^T$  and  $\mathbf{u} \mathbf{e}_i^T + \sum_{m=1, m \neq i}^n \mathbf{x}_m \mathbf{e}_m^T$ .

*Proof.* Let  $\mathbf{u}, \mathbf{v} \in R^n \setminus J(R)^n$ . Then we obtain the following system of matrix products in Equation (18):

$$\begin{aligned} \mathbf{u} \mathbf{e}_i^T \left( \mathbf{e}_i \mathbf{v}^T + \sum_{m=1, m \neq i}^n \mathbf{e}_m \mathbf{x}_m^T \right) &= \mathbf{z} \mathbf{e}_i^T; \\ \mathbf{v} \mathbf{e}_i^T \left( \mathbf{e}_i \mathbf{u}^T + \sum_{m=1, m \neq i}^n \mathbf{e}_m \mathbf{x}_m^T \right) &= \mathbf{y} \mathbf{e}_i^T; \\ \mathbf{e}_i \mathbf{u}^T \left( \mathbf{v} \mathbf{e}_i^T + \sum_{m=1, m \neq i}^n \mathbf{x}_m \mathbf{e}_m^T \right) &= \mathbf{e}_i \mathbf{z}^T; \\ \mathbf{e}_i \mathbf{v}^T \left( \mathbf{u} \mathbf{e}_i^T + \sum_{m=1, m \neq i}^n \mathbf{x}_m \mathbf{e}_m^T \right) &= \mathbf{e}_i \mathbf{y}^T, \end{aligned} \quad (18)$$

with  $\mathbf{e}_i^T \mathbf{z} = \mathbf{e}_i^T \mathbf{y} = \mathbf{z}^T \mathbf{e}_i = \mathbf{y}^T \mathbf{e}_i = \mathbf{u}^T \mathbf{v}$ .

Since  $I_R - \mathbf{u}^T \mathbf{v} \notin U(R)$  and  $I_R - \mathbf{v}^T \mathbf{u} \notin U(R)$ , the equivalent conditions for the invertibility of these matrices can be expressed through Equation (19):

$$\begin{aligned} I_n - \mathbf{z} \mathbf{e}_i^T \notin U(M_n(R)) &\iff I_R - \mathbf{e}_i^T \mathbf{z} \notin U(R); \\ I_n - \mathbf{y} \mathbf{e}_i^T \notin U(M_n(R)) &\iff I_R - \mathbf{e}_i^T \mathbf{y} \notin U(R); \\ I_n - \mathbf{e}_i \mathbf{z}^T \notin U(M_n(R)) &\iff I_R - \mathbf{z}^T \mathbf{e}_i \notin U(R); \\ I_n - \mathbf{e}_i \mathbf{y}^T \notin U(M_n(R)) &\iff I_R - \mathbf{y}^T \mathbf{e}_i \notin U(R). \end{aligned} \quad (19)$$

$\square$

We illustrate Corollary 3.22 above with the following example.

**Example 3.23.** In Figure 1, we have a 2-array Jacobson graph of  $\mathbb{Z}_2$ .

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{-----} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{-----} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Figure 1. Graph  $\mathfrak{J}_{\mathbb{Z}_2}^2$

By Corollary 3.22, we can determine the set of matrices that are adjacent to  $\mathbf{e}_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix}^T = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$  in graph  $\mathfrak{J}_{M_2(\mathbb{Z}_2)}$  by Figure 2.

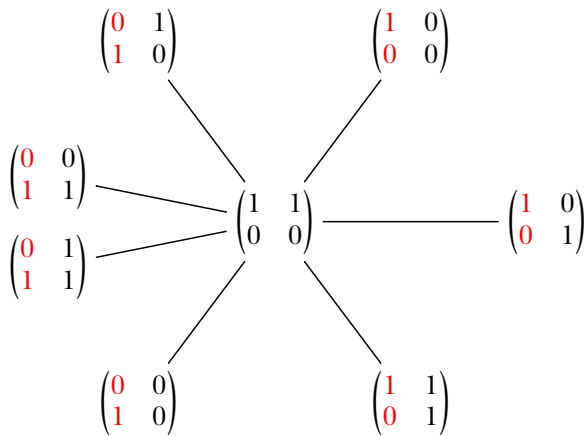


Figure 2. All Matrices which are Adjacent to  $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$  in  $\mathfrak{J}_{M_2(\mathbb{Z}_2)}$

#### 4. CONCLUSIONS

This paper provides a clear structural explanation for the existence of subgraphs induced by subrings within Jacobson-type graphs. Our findings demonstrate that the finiteness condition plays a crucial role in preserving the inclusion relations among units and radicals. Moreover, we have completely characterized the conditions under which the various Jacobson-type graphs associated with a ring are isomorphic. The finiteness assumption is essential to both main results, as these embeddings and structural properties may fail for infinite rings. Natural directions for further research include extending these results to infinite or non-commutative rings and investigating how other graph products interact with Jacobson-type constructions.

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