

Independence and Matchings on Cayley Digraphs of Transformation Semigroups with Fixed Sets

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Abstract

Let Y be a nonempty set of X and denote by $Fix(X, Y)$ a semigroup of full transformations α on X provided that $a\alpha = a$ for all $a \in Y$. Further, let A be a nonempty subset of $Fix(X, Y)$. The Cayley digraph $\text{Cay}(Fix(X, Y), A)$ is defined to be a digraph whose vertex set is $Fix(X, Y)$ and arc set is the set of all ordered pairs (α, β) ; where $\alpha, \beta \in Fix(X, Y)$, such that $\beta = \alpha\gamma$ for some $\gamma \in A$. A nonempty subset I of $Fix(X, Y)$ is said to be independent if every two vertices in I are independent, that is, there is no arc joining them. The maximum cardinality of an independent set is called the independence number. Moreover, an edge independent set is called a matching and the maximum cardinality of an edge independent set is called the matching number. In this paper, we determine several independence and matching parameters of Cayley digraphs of $Fix(X, Y)$ whose connection sets are relative to minimal idempotents and permutations.

Keywords

Cayley Digraphs, Transformation Semigroups with Fixed Sets, Independence, Matchings, Minimal Idempotents, Permutations

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1. INTRODUCTION

An interesting object used to present a connection between algebras and graph theory is known as the Cayley digraph which is firstly constructed from a finite group. The investigation of Cayley graphs provides a vital bridge between abstract algebra and structural graph theory, allowing algebraic properties of algebraic structures to be visualized and analyzed through combinatorial topologies. While Cayley graphs of groups have been extensively studied due to their high degree of symmetry and vertex-transitivity, Cayley digraphs of semigroups, especially transformation semigroups, present a richer and far more complex structural landscape defined by directional flows and inherent asymmetry.

Throughout the paper, all sets are assumed to be finite. Let S be a semigroup and A a nonempty subset of S . The Cayley digraph $\text{Cay}(S, A)$ of a semigroup S with respect to a connection set A is defined to be a digraph with vertex set S and arc set consisting of all ordered pairs (x, xa) for some $a \in A$ and x is an arbitrary element in S . The Cayley digraphs have been investigated for various types of groups and semigroups, including important applications, see for instance (Feng et al., 2022; Ilić-Georgijević, 2023; Kelarev, 2003; Khosravi et al., 2021; Knauer and Knauer, 2019; Limkul and Panma, 2023; Ma et al.,

2022; Nupo and Chaiya, 2023) and references therein. One of the well-known semigroups is related to transformations of a set. In this paper, we consider a semigroup of transformations with fixed sets. Let $T(X)$ be a semigroup of transformations on X under composition and called the *full transformation semigroup*. In 2013, Honyam and Sanwong (Honyam and Sanwong, 2013) defined $Fix(X, Y)$ as follows:

$$Fix(X, Y) = \{\alpha \in T(X) : a\alpha = a \text{ for all } a \in Y\}.$$

Indeed, $Fix(X, Y)$ is a subsemigroup of $T(X)$ which is called *transformation semigroup with fixed set*. They introduced the regularity, Green's relations and ideals of $Fix(X, Y)$. Later, Chaiya et al. (2016) proposed the concept of the natural partial order on $Fix(X, Y)$. The literature review of researches related to Cayley digraphs of transformation semigroups can be found in (Riyas and Geetha, 2018; Tisklang and Panma, 2018). Nupo and Pookpienlert (2020) studied the structural properties of Cayley digraphs of $Fix(X, Y)$. Later in 2021, they (Nupo and Pookpienlert, 2021) investigated the domination parameters on Cayley digraphs of $Fix(X, Y)$ with respect to minimal idempotents and permutations.

The well-known parameters we consider in this research are independence parameters and matching parameters of Cayley digraphs of $Fix(X, Y)$. These parameters are known as

the graph invariant property. In graph theory, studying graph invariants is the prominent research area which plays a role for applying results to approximate some parameters in real world models. One of such invariants is an independence number of graphs or digraphs. It is the size of a maximum independent set in graphs. Many researchers are interested to find a maximum independent set called the maximum independent set problem which is an NP-hard optimization problem. In addition, the independence number of several types of graphs or digraphs are spaciouly investigated. For example, Frieze (1990) determined the independence number of random graphs. Selkow (1993) obtained the independence number of graphs in terms of degrees. Csizmadia (1998) showed some results about the independence number of minimum distance graphs. Lichiardopol (2005) determined the independence number of iterated line digraphs. Asmerom et al. (2011) studied the independence number of the cartesian product of graphs. Panma and Nupo (2018) investigated the independence number of Cayley digraphs of rectangular groups. Later, Brešar and Pabon (2019) studied the independence number of products of Kneser graphs.

For the part of the matching number, Henning and Yeo (2006) determined the matching number of claw-free graphs. Further, Henning et al. (2008) proposed lower and upper bounds for matching numbers of regular graphs. Vukičević (2011) presented methods for determination of the existence and enumeration of perfect matchings and provided applications to chemistry. Ciucu et al. (2012) considered the perfect matchings on fisher graphs of cubic graphs and gave several applications including two concrete three-dimensional crystal structures whose entropy can be computed exactly using their results. Dong et al. (2013) computed the number of perfect matchings of line graphs and showed applications in statistical physics. Aaghabali et al. (2015) presented upper bounds for the number of perfect matchings and discrete 2-factor in graphs with given number of vertices and edges. Yang and Xie (2016) developed an efficient approach to identify maximum matchings of a digraph based on the largest geometric multiplicity. Lukotka and Rollová (2017) provided the existence conditions for perfect matchings of regular bipartite graphs. Ye (2018) studied the maximum matchings in regular graphs. Recently, Cseh and Kavitha (2021) constructed the popular matchings in complete graphs and showed that the problem they considered becomes NP-complete.

Within this framework, the transformation semigroup $Fix(X, Y)$, consisting of full transformations on a set X that fix a non-empty subset $Y \subseteq X$, represents a fundamental class of restricted transformation semigroups. Investigating the independence and matching parameters of $Cay(Fix(X, Y), A)$ relative to connection sets A formed by minimal idempotents and permutations is highly motivated by several key theoretical perspectives. Idempotents (particularly minimal idempotents) and permutations constitute the structural backbone of semigroup theory. Minimal idempotents generate the minimal ideals (the kernel) of a semigroup, while permutations characterize its

maximal subgroup structures. Choosing connection sets based on these specific algebraic elements allows us to directly translate core algebraic properties, such as structural idempotency, projectivity, and fixed-point invariance, into graph-theoretic behaviors. Evaluating digraph invariants under these generators exposes how the algebraic constraints on Y influence the overall system dynamics of the semigroup. The independence number and matching number are among the most celebrated structural invariants in graph theory, serving as foundational metrics for network sparsity, internal stability, and edge-disjoint matchings. Determining these exact values for directed graphs is notoriously difficult due to the lack of undirected symmetry. In semigroup Cayley digraphs, where vertex-transitivity generally fails, exact closed-form expressions for independence and matching numbers reveal crucial information regarding non-interfering transition states (independent sets) and maximal concurrent reachability paths (matchings) within asymmetric directed systems. By bridging these two domains, determining these new independence and matching parameters not only advances the structural characterization of transformation semigroups, but also enriches the broader graph-theoretic catalog of exact parameters for directed algebraic networks.

2. EXPERIMENTAL SECTION

We now provide some preliminaries and relevant notations related to Cayley digraphs of semigroups, a class of independence and matchings in digraphs. Further information about digraphs can be found in Pirzada (2012) and others on semigroups can be seen in Clifford and Preston (1961/1967); Howie (1995).

For a digraph D with vertex set $V(D)$ and arc set $E(D)$, let u, v be a pair of distinct vertices of D and I a nonempty subset of $V(D)$. We present definitions for several types of independence of digraphs as follows.

The vertex u is said to be *independent* to v , or simply called u and v are independent, if $(u, v), (v, u) \notin E(D)$. If vertices in I are pairwise independent, then I is called an *independent set* of D . The maximum cardinality of an independent set of D is called the *independence number* of D and denoted by $\alpha(D)$. An independent set with cardinality $\alpha(D)$ is called an α -set of D .

The vertex u is said to be *weakly independent* to v , or simply called u and v are weakly independent, if $(u, v) \notin E(D)$ or $(v, u) \notin E(D)$. If vertices in I are pairwise weakly independent, then I is called a *weakly independent set* of D . The maximum cardinality of a weakly independent set of D is called the *weakly independence number* of D and denoted by $\alpha_w(D)$. A weakly independent set with cardinality $\alpha_w(D)$ is called an α_w -set of D .

The vertex u is said to be *path independent* to v , or simply called u and v are path independent, if u and v are not connected by a dipath in D . That is, a $[u, v]$ -dipath and a $[v, u]$ -dipath do not exist in D where the notation $[u, v]$ -dipath means the dipath joining from u to v in D . If vertices in I are pairwise path independent, then I is called a *path independent set* of D . The maximum cardinality of a path independent set of D is called the *path independence number* of D and denoted by $\alpha_p(D)$. A

path independent set with cardinality $\alpha_p(D)$ is called an α_p -set of D .

The vertex u is said to be *weakly path independent* to v , or simply called u and v are weakly path independent, if there is no dipath joining from u to v or there is no dipath joining from v to u . If vertices in I are pairwise weakly path independent, then I is called a *weakly path independent set* of D . The maximum cardinality of a weakly path independent set of D is called the *weakly path independence number* of D and denoted by $\alpha_{wp}(D)$. A weakly path independent set with cardinality $\alpha_{wp}(D)$ is called an α_{wp} -set of D .

Generally, every (path) independent set of D is also a weakly (path) independent set of D . For more details, if we let X be an independent set of D such that $|X| = \alpha(D)$, then X is a weakly independent set of D . By the definition of $\alpha_w(D)$, it is the maximum cardinality of a weakly independent set of D , so $|X| \leq \alpha_w(D)$. Hence $\alpha(D) \leq \alpha_w(D)$. Similarly, we can conclude that $\alpha_p(D) \leq \alpha_{wp}(D)$.

Moreover, the definition of a matching in a digraph D with vertex set $V(D)$ and arc set $E(D)$ are prescribed as follows. Let M be a nonempty subset of $E(D)$ without loops. The set M is called a *strong matching* or simply called a matching in D if there is no two arcs in M shared vertices in common. Equivalently, we can say that M is a matching in D if every two arcs (a, b) and (c, d) in M are independent, that is, $a, b \notin \{c, d\}$. Moreover, the set M is called a *weakly matching* in D if every two arcs (a, b) and (c, d) in M are weakly independent, that is, $a \neq c$ and $b \neq d$. The maximum cardinality of a (weakly) matching in D is called the (*weakly*) *matching number* of D and denoted by $(\alpha'_w(D))$ $\alpha'(D)$. A (weakly) matching in D with cardinality $(\alpha'_w(D))$ $\alpha'(D)$ is called an $(\alpha'_w$ -set) α' -set of D .

For a digraph with vertex set V and arc set E , we define

$$N^+(v) = \{u \in V \setminus \{v\} : (v, u) \in E\} \text{ and } \\ N^-(v) = \{u \in V \setminus \{v\} : (u, v) \in E\} \text{ for any } v \in V.$$

Now, we introduce some basic concepts of the semigroup $Fix(X, Y)$. Throughout the paper, let $Y = \{a_i : i \in I\}$ be a nonempty fixed subset of X . Further, let $\alpha \in Fix(X, Y)$. Then $a_i\alpha = a_i$ for all $i \in I$. Denote by $X\alpha = \{a_i : i \in I\} \cup \{b_j : j \in J\}$ the image of α . In general, the index set J can be empty which means that $X\alpha = \{a_i : i \in I\} = Y$. Hence we can write α as follows:

$$\alpha = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}$$

such that $A_i = a_i\alpha^{-1}$ and $B_j = b_j\alpha^{-1}$ for each $i \in I$ and $j \in J$. Also, we can observe that $A_i \cap Y = \{a_i\}$ and $B_j \subseteq X \setminus Y$ for all $i \in I$ and $j \in J$.

Next, we present certain subsemigroups of $Fix(X, Y)$ consisting of the set E_m of all minimal idempotents, and the group H_{id_X} of permutations. It has been proved that

$$E_m = \left\{ \begin{pmatrix} A_i \\ a_i \end{pmatrix} : \{A_i : i \in I\} \text{ is a partition of } X \text{ with } a_i \in A_i \right\}$$

which is shown in [Honyam and Sanwong \(2013\)](#). Moreover, it can be verified that α is a left zero element for all $\alpha \in E_m$ and H_{id_X} is the Green's $\mathcal{H}$ -class of $Fix(X, Y)$ containing the identity map id_X .

Clearly, $Fix(X, Y) = \{id_X\}$ whenever $X = Y$. In the sequel, we then consider the case $Y \subseteq X$. Let Γ be the Cayley digraph $\text{Cay}(Fix(X, Y), A)$ of $Fix(X, Y)$ with a nonempty connection set A . For information about Γ not mentioned in this paper, we refer to [Nupo and Pookpienlert \(2020\)](#).

We now recall the following known results, which are helpful for verifying the independence and matchings of Γ .

Theorem 2.1. ([Nupo and Pookpienlert, 2021](#)) The Cayley digraph Γ is the disjoint union of subdigraphs induced by E_m , H_{id_X} and $Fix(X, Y) \setminus (E_m \cup H_{id_X})$, respectively if and only if $A \subseteq H_{id_X}$.

Lemma 2.2. ([Nupo and Pookpienlert, 2020](#)) Let A be a nonempty subset of $Fix(X, Y)$ and μ a vertex of Γ . If $\mu \in E_m$, then $N^+(\mu) = \emptyset$.

Lemma 2.3. ([Nupo and Pookpienlert, 2020](#)) Let A be a nonempty subset of $Fix(X, Y)$. Then Γ is locally connected if and only if $A \subseteq H_{id_X}$.

The following notation table collects mathematical symbols which are frequently used in the paper.

Table 1. Table of Symbols

Symbols	Meanings
Γ	the Cayley digraph of $Fix(X, Y)$
E_m	the set of all minimal idempotents
H_{id_X}	the group of permutations
G	the set $Fix(X, Y) \setminus (E_m \cup H_{id_X})$
$\mathcal{R}$ -class	the equivalence class of Green's $\mathcal{R}$ -relation
$G/\mathcal{R}$	the set of all $\mathcal{R}$ -classes of G
α	the independence number
α_p	the path independence number
α_w	the weakly independence number
α_{wp}	the weakly path independence number
α'	the matching number
α'_w	the weakly matching number

3. RESULTS AND DISCUSSION

3.1 Independence Parameters of Γ Related to Minimal Idempotents

In this section, we propose some results about independence parameters of Γ whose connection sets are contained in the set E_m of minimal idempotents in $Fix(X, Y)$.

Lemma 3.1. Let $A \subseteq E_m$. Then E_m and $Fix(X, Y) \setminus E_m$ are independent sets of Γ .

Proof. We first show that E_m is independent in Γ . Since every minimal idempotent β is a left zero element in Γ , we have $N^+(\beta) = \emptyset$. Hence E_m is an independent set of Γ . Next,

for each $\gamma \in \text{Fix}(X, Y) \setminus E_m$, we obtain that $\gamma\lambda \in E_m$ for all $\lambda \in A \subseteq E_m$. Thus $\text{Fix}(X, Y) \setminus E_m$ is an independent set of Γ .

As a consequence of Lemma 3.1, we obtain a lower bound of $\alpha(\Gamma)$ and show that such a lower bound is sharp.

Theorem 3.2. Let $A \subseteq E_m$. If $M = \{\beta \in E_m : (X \setminus Y)\beta \cap \bigcup_{\alpha \in A} (X \setminus Y)\alpha = \emptyset\}$, then $\alpha(\Gamma) \geq \max\{|E_m|, |\text{Fix}(X, Y) \setminus E_m| + |M|\}$.

Proof. Let $M = \{\beta \in E_m : (X \setminus Y)\beta \cap \bigcup_{\mu \in A} (X \setminus Y)\mu = \emptyset\}$. We will prove that $(\text{Fix}(X, Y) \setminus E_m) \cup M$ is an independent set of Γ . It is clear when $M = \emptyset$. We now consider $M \neq \emptyset$. Let $\beta, \delta \in (\text{Fix}(X, Y) \setminus E_m) \cup M$. By Lemma 3.1, it is known that E_m and $\text{Fix}(X, Y) \setminus E_m$ are independent sets of Γ . Since $M \subseteq E_m$, we obtain that M is also an independent set of Γ . Hence it suffices to consider the case $\beta \in \text{Fix}(X, Y) \setminus E_m$ and $\delta \in M$. Suppose to the contrary that β and δ are not mutually independent. We have mentioned in the proof of Lemma 3.1 that $N^+(\mu) = \emptyset$ for every minimal idempotent $\mu \in E_m$, that is, there is no a directed edge from μ to other vertices. In this situation, $\beta \in \text{Fix}(X, Y) \setminus E_m$ and $\delta \in M \subseteq E_m$, it follows that Γ does not contain an arc of the form (δ, β) , that is, $(\delta, \beta) \notin E(\Gamma)$. Therefore, we consider only the case $(\beta, \delta) \in E(\Gamma)$. Then there exists $\lambda \in A$ such that $\delta = \beta\lambda$. Since $\beta \in \text{Fix}(X, Y) \setminus E_m$, there exists $x \in X \setminus Y$ for which $x\beta \in X \setminus Y$. Thus $(x\beta)\lambda = x\beta\lambda = x\delta \in (X \setminus Y)\delta \cap \bigcup_{\mu \in A} (X \setminus Y)\mu$, which contradicts $\delta \in M$. Therefore, $(\text{Fix}(X, Y) \setminus E_m) \cup M$ is an independent set of Γ .

Proposition 3.3. The lower bound presented in Theorem 3.2 is sharp.

Proof. Let $|X \setminus Y| = 1$. Then $\text{Fix}(X, Y) = E_m \cup \{id_X\}$, which implies that $|E_m| > |\text{Fix}(X, Y) \setminus E_m| + |M|$. Since E_m is independent as shown in Lemma 3.1, we have $\alpha(\Gamma) \geq |E_m|$. If there exists an independent set I of Γ in which $|I| > |E_m|$, then $I = \text{Fix}(X, Y)$. However, $\text{Fix}(X, Y)$ is not independent since $(id_X, \lambda) \in E(\Gamma)$ for some $\lambda \in A$, a contradiction. Therefore,

$$\alpha(\Gamma) = |E_m| = \max\{|E_m|, |\text{Fix}(X, Y) \setminus E_m| + |M|\}.$$

Further, let $|X \setminus Y| > |Y|$, $|A| = 1$, and $\bigcup_{\mu \in A} (X \setminus Y)\mu \subseteq Y$. So $M \neq \emptyset$ and $|\text{Fix}(X, Y) \setminus E_m| \geq |X \setminus Y|^{X|Y|} > |Y|^{X|Y|} = |E_m|$. By Lemma 3.1, we have $(\text{Fix}(X, Y) \setminus E_m) \cup M$ is an independent set of Γ . Therefore, $\alpha(\Gamma) \geq |\text{Fix}(X, Y) \setminus E_m| + |M|$. Suppose that there exists an independent set I of Γ in which $\alpha(\Gamma) = |I| > |\text{Fix}(X, Y) \setminus E_m| + |M|$. Then $I \cap (E_m \setminus M) \neq \emptyset$. Now, we have two possibilities to consider.

Case 1: $|N^-(\beta)| = 1$ for all $\beta \in I \cap (E_m \setminus M)$. Let $J = I \cap (E_m \setminus M)$. Define $K = (I \setminus J) \cup \bigcup_{\beta \in J} N^-(\beta)$. We show that $K \subseteq (\text{Fix}(X, Y) \setminus E_m) \cup M$. Let $\beta \in K$. If $\beta \in I \setminus J$, then either $\beta \in \text{Fix}(X, Y) \setminus E_m$ or $\beta \in M$. It follows that $\beta \in (\text{Fix}(X, Y) \setminus E_m) \cup M$. If $\beta \in N^-(\lambda)$ for some $\lambda \in J$, then $(\beta, \lambda) \in E(\Gamma)$ and $\lambda \in E_m \setminus M$. Hence $\lambda = \beta\mu$ for some $\mu \in A \subseteq E_m$. Suppose to the contrary that $\beta \in E_m$. Thus $\lambda = \beta\mu = \beta$ since β is a left zero element in $\text{Fix}(X, Y)$. This contradicts $\beta \in N^-(\lambda)$. Therefore, $\beta \in \text{Fix}(X, Y) \setminus E_m$. Consequently, $K \subseteq (\text{Fix}(X, Y) \setminus E_m) \cup M$. Since $(\text{Fix}(X, Y) \setminus E_m) \cup M$ is independent, it implies that K is also independent. Furthermore,

we can observe that, for each $\beta \in J$, $|N^-(\beta)| = 1$ since $|A| = 1$. This yields that $|J| = \sum_{\beta \in J} |N^-(\beta)| = |\bigcup_{\beta \in J} N^-(\beta)|$, which leads to $|K| = |(I \setminus J) \cup \bigcup_{\beta \in J} N^-(\beta)| = |I|$. We obtain that $|I| = |K| \leq |\text{Fix}(X, Y) \setminus E_m| + |M|$, a contradiction.

Case 2: there exists $\beta \in I \cap (E_m \setminus M)$ such that $|N^-(\beta)| \geq 2$. Then $|(I \setminus \{\beta\}) \cup N^-(\beta)| > |I|$. To show that $(I \setminus \{\beta\}) \cup N^-(\beta)$ is an independent set of Γ , let $\lambda, \mu \in (I \setminus \{\beta\}) \cup N^-(\beta)$. It is clear that λ is independent to μ if both of them belong to $I \setminus \{\beta\}$ since I is independent. If $\lambda, \mu \in N^-(\beta)$, then $\lambda, \mu \in \text{Fix}(X, Y) \setminus E_m$ by the argument mentioned in Case 1. Since $\text{Fix}(X, Y) \setminus E_m$ is independent by Lemma 3.1, λ is independent to μ . If $\lambda \in I \setminus \{\beta\}$ and $\mu \in N^-(\beta)$, then $(\mu, \beta) \in E(\Gamma)$. From $|A| = 1$, it follows that $N^+(\mu) = \{\beta\}$. Since $\lambda \neq \beta$, we get that λ is independent to μ . Therefore, $(I \setminus \{\beta\}) \cup N^-(\beta)$ is an independent set of Γ , which contradicts the maximality of I .

Consequently, we obtain $\alpha(\Gamma) = |\text{Fix}(X, Y) \setminus E_m| + |M|$. Moreover, $\alpha(\Gamma) = \max\{|E_m|, |\text{Fix}(X, Y) \setminus E_m| + |M|\}$.

The following theorem presents the exact value of the weakly independence number of Γ .

Theorem 3.4. Let $A \subseteq E_m$. Then $\alpha_w(\Gamma) = |X|^{X|Y|}$.

Proof. Since every arc (except loops) of Γ is precisely expressed in the form (β, δ) where $\beta \in \text{Fix}(X, Y) \setminus E_m$ and $\delta \in E_m$, this implies that $\text{Fix}(X, Y)$ is weakly independent. Therefore, $\alpha_w(\Gamma) = |\text{Fix}(X, Y)| = |X|^{X|Y|}$.

We now investigate the properties of other independence parameters of Γ including path independence and weakly path independence. In this situation, a connection set of Γ is contained in E_m . Thus the general structure of Γ is provided by Lemma 2.2 as follows. All minimal idempotents in $\text{Fix}(X, Y)$ have loops attached to themselves and further, $N^+(\beta) = \emptyset$ for all $\beta \in E_m$. Moreover, all arcs (except loops) of Γ are precisely expressed as (λ, μ) where $\lambda \in \text{Fix}(X, Y) \setminus E_m$ and $\mu \in E_m$. This also implies that $N^-(\delta) = \emptyset$ for all $\delta \in \text{Fix}(X, Y) \setminus E_m$. Hence there are no two distinct elements β, δ in $\text{Fix}(X, Y)$ satisfying the following conditions:

- (i) $(\beta, \delta), (\delta, \beta) \in E(\Gamma)$;
- (ii) $[\beta, \delta]$ -dipath and $[\delta, \beta]$ -dipath exist in Γ .

Therefore, $\text{Fix}(X, Y)$ is weakly independent and weakly path independent in Γ , respectively. In addition, if $[\beta, \delta]$ -dipath exists in Γ , then such dipath would be of length 1, that is, the arc (β, δ) , immediately. Consequently, we obtain the following theorem.

Theorem 3.5. If $A \subseteq E_m$, then the following statements hold:

- (i) $\alpha_{wp}(\Gamma) = \alpha_w(\Gamma)$ and
- (ii) $\alpha_p(\Gamma) = \alpha(\Gamma)$.

3.2 Independence Parameters of Γ Related to Permutations

According to Theorem 2.1, the Cayley digraph Γ is the disjoint union of subdigraphs induced by E_m, H_{id_X} and $\text{Fix}(X, Y) \setminus (E_m \cup H_{id_X})$ if and only if $A \subseteq H_{id_X}$. For convenience, we denote by $\Gamma[S]$ the subdigraph of Γ induced from S where S is a subset of $\text{Fix}(X, Y)$. In order to investigate the independence properties of Γ with respect to a connection set $A \subseteq H_{id_X}$, we then separate our study into three parts corresponding to E_m, H_{id_X} and $\text{Fix}(X, Y) \setminus (E_m \cup H_{id_X})$, respectively.

3.2.1 Independence Parameters of $\Gamma[E_m]$

Theorem 3.6. Let $A \subseteq H_{id_X}$ be a connection set of Γ . Then $\alpha(\Gamma[E_m]) = |E_m|$.

Proof. We first show that $N^+(\mu) = \emptyset = N^-(\mu)$ for all $\mu \in E_m$. Let $\mu \in E_m$. We have by Lemma 2.2 that $N^+(\mu) = \emptyset$. Consider the in-neighborhood of μ , that is, $N^-(\mu) = \{\rho \in \text{Fix}(X, Y) \setminus \{\mu\} : (\rho, \mu) \in E(\Gamma)\}$.

If there exists $\rho \in N^-(\mu)$, then $\rho \neq \mu$ and $\mu = \rho\lambda$ for some $\lambda \in A \subseteq H_{id_X}$. Let $x \in X$. If $x\rho \notin Y$, then $x\mu = x(\rho\lambda) = (x\rho)\lambda \notin Y$, which is impossible. Therefore, $x\rho \in Y$ and thus $x\mu = (x\rho)\lambda = x\rho$. Consequently, $\mu = \rho$, which is a contradiction. Hence $N^-(\mu) = \emptyset$. Therefore, every minimal idempotent of $\text{Fix}(X, Y)$ is an isolated loop. This implies that E_m is an independent set of $\Gamma[E_m]$, which leads to $\alpha(\Gamma[E_m]) = |E_m|$ immediately.

The proof of Theorem 3.6 related to the independence number of $\Gamma[E_m]$ can be applied to the weakly independence number $\alpha_w(\Gamma[E_m])$, the path independence number $\alpha_p(\Gamma[E_m])$, and the weakly path independence number $\alpha_{wp}(\Gamma[E_m])$. Thus we directly obtain the following theorem.

Theorem 3.7. Let $A \subseteq H_{id_X}$ be a connection set of Γ . Then $\alpha(\Gamma[E_m]) = \alpha_w(\Gamma[E_m]) = \alpha_p(\Gamma[E_m]) = \alpha_{wp}(\Gamma[E_m])$.

3.2.2 Independence Parameters of $\Gamma[H_{id_X}]$

We start this part with the characterization of $\Gamma[H_{id_X}]$ with respect to a connection set $A \subseteq H_{id_X}$.

Theorem 3.8. Let $p = \frac{|H_{id_X}|}{|\langle A \rangle|}$ and $H_{id_X}/\langle A \rangle = \{\lambda_1\langle A \rangle, \lambda_2\langle A \rangle, \dots, \lambda_p\langle A \rangle\}$ be the set of all left cosets of $\langle A \rangle$ in H_{id_X} . Then $\Gamma[H_{id_X}]$ is the disjoint union of isomorphic subdigraphs $\Gamma[\lambda_i\langle A \rangle]$ where $\Gamma[\lambda_i\langle A \rangle]$ is a connected component of $\Gamma[H_{id_X}]$ for all $i \in \{1, 2, \dots, p\}$.

Proof. Obviously, $V(\Gamma[H_{id_X}]) = H_{id_X} = \bigcup_{i=1}^p V(\Gamma[\lambda_i\langle A \rangle])$ and $\Gamma[\lambda_i\langle A \rangle]$ is a subdigraph of $\Gamma[H_{id_X}]$. Next, let $(\lambda, \delta) \in E(\Gamma[H_{id_X}])$. Then there exists $\mu \in A$ such that $\delta = \lambda\mu$. Assume that $\lambda \in \lambda_k\langle A \rangle$ for some $k \in \{1, 2, \dots, p\}$. Hence $\lambda = \lambda_k\varepsilon_1\varepsilon_2 \dots \varepsilon_q$ for some $\varepsilon_i \in A$ where $i \in \{1, 2, \dots, q\}$. Consider $\delta = \lambda\mu = \lambda_k\varepsilon_1\varepsilon_2 \dots \varepsilon_q\mu \in \lambda_k\langle A \rangle$. We obtain that $(\lambda, \delta) \in E(\Gamma[\lambda_k\langle A \rangle])$. This implies that $\Gamma[H_{id_X}]$ is the disjoint union of subdigraphs $\Gamma[\lambda_i\langle A \rangle]$ for all $i \in \{1, 2, \dots, p\}$.

For each $t \in \{1, 2, \dots, p\}$, we now show that $\Gamma[\lambda_t\langle A \rangle]$ is connected. Let $\lambda, \delta \in \lambda_t\langle A \rangle$. Then $\lambda = \lambda_t\varepsilon_1\varepsilon_2 \dots \varepsilon_q$ and $\delta = \lambda_t\varepsilon'_1\varepsilon'_2 \dots \varepsilon'_r$ for some $\varepsilon_i, \varepsilon'_j \in A$ where $i \in \{1, 2, \dots, q\}$ and $j \in \{1, 2, \dots, r\}$. Since $A \subseteq H_{id_X}$, so $\varepsilon_i^{-1} \in \langle A \rangle \subseteq H_{id_X}$ for all $i \in \{1, 2, \dots, q\}$ and thus $\lambda_i = \lambda\varepsilon_q^{-1} \dots \varepsilon_2^{-1}\varepsilon_1^{-1}$. Hence $\delta = \lambda\varepsilon_q^{-1} \dots \varepsilon_2^{-1}\varepsilon_1^{-1}\varepsilon'_1\varepsilon'_2 \dots \varepsilon'_r \in \lambda\langle A \rangle$, which implies that $\delta = \lambda\mu_1\mu_2 \dots \mu_k$ for some $\mu_1, \mu_2, \dots, \mu_k \in A$. That means $\lambda, \lambda\mu_1, \lambda\mu_1\mu_2, \dots, \lambda\mu_1\mu_2 \dots \mu_k = \delta$ form a dipath from λ through to δ in $\Gamma[\lambda_t\langle A \rangle]$. Therefore, $\Gamma[\lambda_t\langle A \rangle]$ is a connected component of $\Gamma[H_{id_X}]$.

Next, let $i, j \in \{1, 2, \dots, p\}$. We prove that $\Gamma[\lambda_i\langle A \rangle] \cong \Gamma[\lambda_j\langle A \rangle]$. Define $\phi : \lambda_i\langle A \rangle \rightarrow \lambda_j\langle A \rangle$ by $\phi(\lambda_i\mu) = \lambda_j\mu$ for all $\mu \in \langle A \rangle$. Since $\lambda_i, \lambda_j \in H_{id_X}$, it follows that ϕ is well-defined and bijective. Let $(\lambda_i\mu_1, \lambda_i\mu_2) \in E(\Gamma[\lambda_i\langle A \rangle])$. Then $\lambda_i\mu_2 = \lambda_i\mu_1\rho$ for some $\rho \in A$. We obtain that $\phi(\lambda_i\mu_2) = \phi(\lambda_i\mu_1\rho) =$

$\lambda_j\mu_1\rho = \phi(\lambda_i\mu_1)\rho$. This yields that $(\phi(\lambda_i\mu_1), \phi(\lambda_i\mu_2)) \in E(\Gamma[\lambda_j\langle A \rangle])$.

Next, let $(\lambda_j\eta_1, \lambda_j\eta_2) \in E(\Gamma[\lambda_j\langle A \rangle])$. Then $\lambda_j\eta_2 = \lambda_j\eta_1\xi$ for some $\xi \in A$, which implies that $\eta_2 = \eta_1\xi$. Since ϕ is surjective, there exist $\lambda_i\eta_1, \lambda_i\eta_2 \in \lambda_i\langle A \rangle$ such that $\phi(\lambda_i\eta_1) = \lambda_j\eta_1$ and $\phi(\lambda_i\eta_2) = \lambda_j\eta_2$. Considering $\phi^{-1}(\lambda_j\eta_2) = \lambda_i\eta_2 = \lambda_i\eta_1\xi = \phi^{-1}(\lambda_j\eta_1)\xi$ where $\xi \in A$, we have $(\phi^{-1}(\lambda_j\eta_1), \phi^{-1}(\lambda_j\eta_2)) \in E(\Gamma[\lambda_i\langle A \rangle])$. Consequently, $\Gamma[\lambda_i\langle A \rangle] \cong \Gamma[\lambda_j\langle A \rangle]$, as required.

As a consequence of Lemma 3.8 and by applying Lemma 5 in Panma and Nupo (2018), we obtain the following result.

Theorem 3.9. Let $A \subseteq H_{id_X}$ be a connection set of Γ . Then

$$\frac{|H_{id_X}|}{|\langle A \rangle|} \left\lceil \frac{|\langle A \rangle|}{2|A| + 1} \right\rceil \leq \alpha(\Gamma[H_{id_X}]) \leq \frac{|H_{id_X}|}{|\langle A \rangle|} \left\lfloor \frac{|\langle A \rangle|}{2} \right\rfloor.$$

Surprisingly, the weakly independence number of Γ can be obtained from its independence number as follows.

Theorem 3.10. Let $A \subseteq H_{id_X}$ and $A^{-1} = \{\lambda^{-1} : \lambda \in A\}$. Then

$$\begin{aligned} \alpha_w(\text{Cay}(H_{id_X}, A)) &= \alpha_w(\text{Cay}(H_{id_X}, A \cap A^{-1})) \\ &= \alpha(\text{Cay}(H_{id_X}, A \cap A^{-1})). \end{aligned}$$

Proof. Since $\text{Cay}(H_{id_X}, A \cap A^{-1})$ is a spanning subdigraph of $\text{Cay}(H_{id_X}, A)$, we have $\alpha_w(\text{Cay}(H_{id_X}, A)) \leq \alpha_w(\text{Cay}(H_{id_X}, A \cap A^{-1}))$. Let I be a weakly independent set of $\text{Cay}(H_{id_X}, A \cap A^{-1})$ such that $|I| = \alpha_w(\text{Cay}(H_{id_X}, A \cap A^{-1}))$. We will show that I is weakly independent in $\text{Cay}(H_{id_X}, A)$. Suppose to the contrary that there exist $\lambda_1, \lambda_2 \in I$ such that they are not weakly independent in $\text{Cay}(H_{id_X}, A)$. That is, $(\lambda_1, \lambda_2), (\lambda_2, \lambda_1) \in E(\text{Cay}(H_{id_X}, A))$ which implies that $\lambda_2 = \lambda_1\mu$ and $\lambda_1 = \lambda_2\rho$ for some $\mu, \rho \in A \subseteq H_{id_X}$, respectively. Thus $\lambda_2 = (\lambda_2\rho)\mu$ which leads to $\rho\mu = id_X$ and hence $\rho, \mu \in A \cap A^{-1}$. Therefore, $(\lambda_1, \lambda_2), (\lambda_2, \lambda_1) \in E(\text{Cay}(H_{id_X}, A \cap A^{-1}))$, a contradiction. So we conclude that $\alpha_w(\text{Cay}(H_{id_X}, A \cap A^{-1})) = |I| \leq \alpha_w(\text{Cay}(H_{id_X}, A))$. Moreover, it is easy to verify that $\alpha_w(\text{Cay}(H_{id_X}, A \cap A^{-1})) = \alpha(\text{Cay}(H_{id_X}, A \cap A^{-1}))$.

3.2.3 Independence Parameters of $\Gamma[\text{Fix}(X, Y) \setminus (E_m \cup H_{id_X})]$

From now on, we denote by G the set $\text{Fix}(X, Y) \setminus (E_m \cup H_{id_X})$. We now recall the definition of the $\mathcal{R}$ -Green's relation on a semigroup introduced by Howie (1995). Let S be a semigroup with identity and $a, b \in S$. Then $a \mathcal{R} b$ if and only if there exist $x, y \in S$ such that $a = bx$ and $b = ay$. Obviously, the relation $\mathcal{R}$ is an equivalence relation on the semigroup S . Further, we denote by R_a the $\mathcal{R}$ -class of S containing a . Honyam and Sanwong (2013) characterized the $\mathcal{R}$ -Green's relation on $\text{Fix}(X, Y)$ as follows. For each $\alpha, \beta \in \text{Fix}(X, Y)$,

$$\alpha \mathcal{R} \beta \quad \text{if and only if} \quad \pi_\alpha = \pi_\beta$$

where $\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$. Therefore, for each $\alpha \in \text{Fix}(X, Y)$, the $\mathcal{R}$ -class of $\text{Fix}(X, Y)$ containing α is

$$R_\alpha = \{\beta \in \text{Fix}(X, Y) : \pi_\beta = \pi_\alpha\}.$$

To consider $\alpha(\Gamma[G])$ and $\alpha_w(\Gamma[G])$, we investigate the following lemmas.

Lemma 3.11. Let $A \subseteq H_{id_X} \setminus \{id_X\}$ be a connection set of Γ . If R is an $\mathcal{R}$ -class of G , then there exists $\lambda \in R$ such that $N^+(\lambda) \neq \emptyset$.

Proof. Let R be an $\mathcal{R}$ -class of G and $\beta \in G$. Then we can write β as follows:

$$\beta = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix}$$

where $0 \neq |J| \neq |X \setminus Y|$. Let $\mu \in A \subseteq H_{id_X} \setminus \{id_X\}$. If $b_j \mu \neq b_j$ for some $j \in J$, then $(\beta, \beta \mu) \in E(\Gamma[G])$ in which $\beta \mu \neq \beta$ and $\beta \mu \mathcal{R} \beta$. Hence, we can choose $\lambda = \beta \mu \in R_\beta$ and then $N^+(\lambda) = N^+(\beta \mu) \neq \emptyset$. If $b_j \mu = b_j$ for all $j \in J$, then we choose $b_k \in (X \setminus Y) \setminus \{b_j : j \in J\}$ such that $b_k \mu \neq b_k$. Define $\lambda \in \text{Fix}(X, Y)$ by

$$\lambda = \begin{pmatrix} A_i & B_{j'} & B_{j_0} \\ a_i & b_{j'} & b_k \end{pmatrix}_{j' \in J \setminus \{j_0\}}.$$

Then $\lambda \in G$ and $(\lambda, \lambda \mu) \in E(\Gamma[G])$. Moreover, we can observe that $\lambda \mu \neq \lambda$ and $\lambda \mathcal{R} \beta$. Therefore, $N^+(\lambda) \neq \emptyset$.

Lemma 3.12. Let $A \subseteq H_{id_X}$ be a connection set of Γ . If $(\alpha, \beta) \in E(\Gamma)$, then $\alpha \mathcal{R} \beta$.

Proof. Let $(\alpha, \beta) \in E(\Gamma)$. Then there exists $\mu \in A \subseteq H_{id_X}$ in which $\beta = \alpha \mu$. Thus $\alpha = \beta \mu^{-1}$, which implies that $\alpha \mathcal{R} \beta$, as required.

We now present the number $|G/\mathcal{R}|$ of all $\mathcal{R}$ -classes of G . Actually, the number $|G/\mathcal{R}|$ relates to the Stirling number $S(n, k)$ of the second kind, which is the number of ways to partition a set of n objects into k nonempty subsets.

Lemma 3.13. Let $|X| = n$ and $|Y| = m$. Then the number of all $\mathcal{R}$ -classes in G is

$$|G/\mathcal{R}| = \sum_{i=1}^{n-m-1} \sum_{j=0}^m \sum_{k=j}^{n-m-1} \binom{m}{j} \binom{n-m}{k} S(k, j) S(n-m-k, i) j!.$$

Proof. In order to determine the number of $\mathcal{R}$ -classes in G , it suffices to count the number of all distinct sets $\pi_\alpha = \{x\alpha^{-1} : x \in X\alpha\}$ in which $\alpha \in G$. Since $G = \text{Fix}(X, Y) \setminus (E_m \cup H_{id_X})$, the partition $\mathcal{P}$ of X (domain of each element in G) must contain m cells of elements in Y and i cells of elements outside Y where $1 \leq i \leq n-m-1$. Indeed, the element $\alpha \in G$ can be written as

$$\alpha = \begin{pmatrix} A_1 & A_2 & \cdots & A_m & B_1 & B_2 & \cdots & B_i \\ a_1 & a_2 & \cdots & a_m & b_1 & b_2 & \cdots & b_i \end{pmatrix}.$$

To investigate the possibilities of $\mathcal{P}$, we may assume that

$$\mathcal{P} = \{A_1, A_2, \dots, A_m, B_1, B_2, \dots, B_i\}.$$

Let j be the number of $A_r \in \mathcal{P}$ such that $A_r \cap (X \setminus Y) \neq \emptyset$. Obviously, $0 \leq j \leq m$. Then the number of ways to construct a subset $\{A_1, A_2, \dots, A_m\}$ of $\mathcal{P}$ is

$$\sum_{j=0}^m \binom{m}{j} j!.$$

Moreover, for each $\{A_1, A_2, \dots, A_m\}$ of $\mathcal{P}$ satisfying such number j , we have to choose elements from $X \setminus Y$ and put them into those j cells, randomly. Then the number of ways in this process is

$$\sum_{k=j}^{n-m-1} \binom{n-m}{k} S(k, j).$$

Hence the number of ways to construct a subset $\{B_1, B_2, \dots, B_i\}$ of $\mathcal{P}$ is the number of ways to distinguish $n-m-k$ elements (remaining from the above process) into those i sets which equals $S(n-m-k, i)$. Consequently, the number of required partitions is equal to

$$\sum_{i=1}^{n-m-1} \left[\sum_{j=0}^m \binom{m}{j} j! \left[\sum_{k=j}^{n-m-1} \binom{n-m}{k} S(k, j) S(n-m-k, i) \right] \right].$$

Recall that two elements $\alpha, \beta \in \text{Fix}(X, Y)$ are $\mathcal{F}$ -related if and only if $|X\alpha| = |X\beta|$, which can be seen in [Honyam and Sanwong \(2013\)](#). We now present the characterization of $\Gamma[G]$.

Lemma 3.14. Let $|X| = n$, $|Y| = m$ and $|X \setminus Y| = r$. If $A \subseteq H_{id_X}$, then the following statements hold.

- (i) If $\lambda, \delta \in G$ and $\lambda \mathcal{F} \delta$, then $\Gamma[R_\lambda] \cong \Gamma[R_\delta]$.
- (ii) $\Gamma[G]$ is the disjoint union of $|G/\mathcal{R}|$ subdigraphs which each of them is induced from an $\mathcal{R}$ -class of G .

Proof. Let $\lambda, \delta \in G$ and $\lambda \mathcal{F} \delta$. Furthermore, let R_λ and R_δ be $\mathcal{R}$ -classes of G containing λ and δ , respectively. We first show that $\Gamma[R_\lambda] \cong \Gamma[R_\delta]$. Since $\lambda \mathcal{F} \delta$, we can write

$$\lambda = \begin{pmatrix} A_i & B_j \\ a_i & b_j \end{pmatrix} \quad \text{and} \quad \delta = \begin{pmatrix} A'_i & B'_j \\ a_i & b'_j \end{pmatrix}.$$

Define $\varphi : R_\lambda \rightarrow R_\delta$ by

$$\varphi \left(\begin{pmatrix} A_i & B_j \\ a_i & m_j \end{pmatrix} \right) = \begin{pmatrix} A'_i & B'_j \\ a_i & m_j \end{pmatrix} \quad \text{for all} \quad \begin{pmatrix} A_i & B_j \\ a_i & m_j \end{pmatrix} \in R_\lambda.$$

It is clear that φ is a well-defined bijection. Now, we prove that φ and φ^{-1} are arc-preserving. Let $(\xi, \rho) \in E(\Gamma[R_\lambda])$. Then $\rho = \xi \mu$ for some $\mu \in A \subseteq H_{id_X}$. So we can write

$$\xi = \begin{pmatrix} A_i & B_j \\ a_i & s_j \end{pmatrix}, \quad \rho = \begin{pmatrix} A_i & B_j \\ a_i & t_j \end{pmatrix}, \quad \mu = \begin{pmatrix} a_i & p_r \\ a_i & p'_r \end{pmatrix},$$

where $r = |X \setminus Y|$. Hence

$$\begin{pmatrix} A_i & B_j \\ a_i & t_j \end{pmatrix} = \begin{pmatrix} A_i & B_j \\ a_i & s_j \end{pmatrix} \begin{pmatrix} a_i & p_r \\ a_i & p'_r \end{pmatrix} = \begin{pmatrix} A_i & B_j \\ a_i & s_j \mu \end{pmatrix},$$

which implies that $t_j = s_j \mu$. Hence

$$\begin{aligned}\varphi(\xi)\mu &= \begin{pmatrix} A'_i & B'_j \\ a_i & s_j \end{pmatrix} \begin{pmatrix} a_i & p_r \\ a_i & p'_r \end{pmatrix} \\ &= \begin{pmatrix} A'_i & B'_j \\ a_i & s_j \mu \end{pmatrix} = \begin{pmatrix} A'_i & B'_j \\ a_i & t_j \end{pmatrix} = \varphi(\rho).\end{aligned}$$

Therefore, $(\varphi(\xi), \varphi(\rho)) \in E(\Gamma[R_\delta])$. On the other hand, assume that $(\xi', \rho') \in E(\Gamma[R_\delta])$. By the surjectivity of φ , there exist $\xi, \rho \in R_\lambda$ such that $\varphi(\xi) = \xi'$ and $\varphi(\rho) = \rho'$. Thus $\rho' = \xi'\eta$, that is, $\varphi(\rho) = \varphi(\xi)\eta$ for some $\eta \in A \subseteq H_{id_X}$. Let

$$\xi = \begin{pmatrix} A_i & B_j \\ a_i & d_j \end{pmatrix}, \quad \rho = \begin{pmatrix} A_i & B_j \\ a_i & c_j \end{pmatrix}, \quad \eta = \begin{pmatrix} a_i & q_r \\ a_i & q'_r \end{pmatrix},$$

where $r = |X \setminus Y|$. Hence

$$\begin{aligned}\begin{pmatrix} A'_i & B'_j \\ a_i & c_j \end{pmatrix} &= \varphi(\rho) = \varphi(\xi)\eta \\ &= \begin{pmatrix} A'_i & B'_j \\ a_i & d_j \end{pmatrix} \begin{pmatrix} a_i & q_r \\ a_i & q'_r \end{pmatrix} \\ &= \begin{pmatrix} A'_i & B'_j \\ a_i & d_j \eta \end{pmatrix}.\end{aligned}$$

This yields that $c_j = d_j\eta$. Therefore,

$$\begin{aligned}\varphi^{-1}(\xi')\eta &= \xi\eta \\ &= \begin{pmatrix} A_i & B_j \\ a_i & d_j \end{pmatrix} \begin{pmatrix} a_i & q_r \\ a_i & q'_r \end{pmatrix} \\ &= \begin{pmatrix} A_i & B_j \\ a_i & d_j \eta \end{pmatrix} = \begin{pmatrix} A_i & B_j \\ a_i & c_j \end{pmatrix} = \rho = \varphi^{-1}(\rho').\end{aligned}$$

Consequently, $(\varphi^{-1}(\xi'), \varphi^{-1}(\rho')) \in E(\Gamma[R_\lambda])$.

Next, we prove that

$$\Gamma[G] = \bigcup \{\Gamma[R] : R \text{ is an } \mathcal{R}\text{-class of } G\}.$$

Obviously, the vertex sets of those two digraphs are equal. Let $(\lambda, \delta) \in E(\Gamma[G])$. Then $\delta = \lambda\mu$ for some $\mu \in A \subseteq H_{id_X}$, which implies that $\lambda = \delta\mu^{-1}$. Hence, $\lambda\mathcal{R}\delta$, that is, $\lambda, \delta \in R_\lambda = V(\Gamma[R_\lambda])$. We conclude that $(\lambda, \delta) \in E(\Gamma[R_\lambda])$. The other containment is clear. Furthermore, the set

$$\{\Gamma[R] : R \text{ is an } \mathcal{R}\text{-class of } G\}$$

is pairwise disjoint since G is partitioned by an equivalence relation $\mathcal{R}$.

The following theorem shows the bounds of $\alpha(\Gamma[G])$ and $\alpha_w(\Gamma[G])$ relative to the number of $\mathcal{R}$ -classes of G and the number of isolated loops in $\Gamma[G]$.

Theorem 3.15. Let $|X| = n$ and $|Y| = m$. If $A \subseteq H_{id_X}$, then

$$|G/\mathcal{R}| + l \leq \alpha(\Gamma[G]) \leq \alpha_w(\Gamma[G]) \leq |G| - \left\lfloor \frac{|A \cap A^{-1}|}{|A|} \right\rfloor |G/\mathcal{R}|,$$

where l is the number of isolated loops in $\Gamma[G]$.

Proof. As we have known that the number of $\mathcal{R}$ -classes of G is equal to $|G/\mathcal{R}|$. By Lemma 3.11, we can choose exactly one vertex which is not an isolated loop from each $\mathcal{R}$ -class of G and let I be the set of such vertices. If there exist two vertices $\lambda_1, \lambda_2 \in I$ such that they are not independent, that is, $(\lambda_1, \lambda_2) \in E(\Gamma[G])$ or $(\lambda_2, \lambda_1) \in E(\Gamma[G])$, this implies by Lemma 3.12 that $\lambda_1\mathcal{R}\lambda_2$, which is a contradiction. Hence, I is an independent set of $\Gamma[G]$. Further, this evidently implies that

$$\alpha(\Gamma[G]) \geq |I| + l = |G/\mathcal{R}| + l.$$

Next, we prove an upper bound of $\alpha_w(\Gamma[G])$. If $A \cap A^{-1} = \emptyset$, then the upper bound is attained. We now consider the case $A \cap A^{-1} \neq \emptyset$. Let K be a subset of G with order greater than

$$|G| - \left\lfloor \frac{|A \cap A^{-1}|}{|A|} \right\rfloor |G/\mathcal{R}|.$$

In this case, it is clear that

$$\left\lfloor \frac{|A \cap A^{-1}|}{|A|} \right\rfloor = 1.$$

We obtain that there exists an $\mathcal{R}$ -class, say R , of G such that $R \subseteq K$. Since $A \cap A^{-1} \neq \emptyset$, there exists $\mu \in A \cap A^{-1}$ which implies that $(\beta, \beta\mu), (\beta\mu, \beta) \in E(\Gamma[G])$ for some $\beta \in R$ and β is not an isolated loop (β always exists by Lemma 3.11). Therefore, K is not a weakly independent set of $\Gamma[G]$. Consequently,

$$\alpha_w(\Gamma[G]) \leq |G| - \left\lfloor \frac{|A \cap A^{-1}|}{|A|} \right\rfloor |G/\mathcal{R}|.$$

Moreover, by the fact that $\alpha(\Gamma[G]) \leq \alpha_w(\Gamma[G])$, the proof is done.

Now, we will present the result on path independence number and weakly path independence number of Γ .

Theorem 3.16. Let $A \subseteq H_{id_X}$ be a connection set of Γ . Then $\alpha_p(\Gamma) = \alpha(\Gamma) = \alpha_{wp}(\Gamma)$.

Proof. It is clear by the definitions of $\alpha(\Gamma)$ and $\alpha_p(\Gamma)$ that $\alpha_p(\Gamma) \leq \alpha(\Gamma)$. Let $\lambda, \delta \in \text{Fix}(X, Y)$. Since $A \subseteq H_{id_X}$, we have by Lemma 2.3 that Γ is locally connected. That is, if a $[\lambda, \delta]$ -dipath exists in Γ , then a $[\delta, \lambda]$ -dipath must exist, as well. Hence the weakly path independence implies the path independence. Thus $\alpha_{wp}(\Gamma) \leq \alpha_p(\Gamma)$. As a consequence of the above argument, we conclude that any two independent vertices must be weakly independent. Therefore, $\alpha(\Gamma) \leq \alpha_{wp}(\Gamma)$. Thus we have $\alpha_{wp}(\Gamma) \leq \alpha_p(\Gamma) \leq \alpha(\Gamma) \leq \alpha_{wp}(\Gamma)$. This completes the proof.

3.3 Matching Parameters of Γ Related to Minimal Idempotents

This section provides results of matching parameters of Γ with respect to E_m . We start with an upper bound of $\alpha(\Gamma)$ which is related to the matching number $\alpha'(\Gamma)$.

Lemma 3.17. Let $A \subseteq E_m$ be a connection set of Γ . Then $\alpha(\Gamma) \leq |X|^{|X|-|Y|} - \alpha'(\Gamma)$.

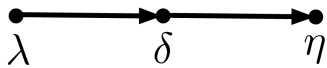


Figure 1. Subdigraph of Γ Induced by (λ, δ) and (δ, η)

Proof. Let T be the maximum matching. Clearly, no more than $|T|$ elements of $V(\Gamma[T])$ can be in an independent set of Γ . Let $V'(\Gamma) = V(\Gamma) \setminus V(\Gamma[T])$. Then $|V'(\Gamma)| + |V(\Gamma[T])| = |V(\Gamma)| = |Fix(X, Y)|$. Therefore, $\alpha(\Gamma) \leq |T| + |V'(\Gamma)| = \alpha'(\Gamma) + |Fix(X, Y)| - |V(\Gamma[T])| = \alpha'(\Gamma) + |Fix(X, Y)| - 2\alpha'(\Gamma) = |Fix(X, Y)| - \alpha'(\Gamma) = |X|^{X|Y|} - \alpha'(\Gamma)$.

We now present the bounds of $\alpha'_w(\Gamma)$ in the following theorem. Indeed, we also obtain that both $\alpha'(\Gamma)$ and $\alpha'_w(\Gamma)$ coincide.

Theorem 3.18. Let $A \subseteq E_m$ be a connection set of Γ . Then

$$\sum_{k=1}^q \binom{q}{k} \binom{r}{k} k!(m-k)^{r-k} \leq \alpha'(\Gamma) = \alpha'_w(\Gamma) \leq |X|^{X|Y|} - \max\{|E_m|, |Fix(X, Y) \setminus E_m| + |M|\},$$

where

$$M = \left\{ \beta \in E_m : (X \setminus Y)\beta \cap \left(\bigcup_{\lambda \in A} (X \setminus Y)\lambda \right) = \emptyset \right\}$$

and $q = \max\{|(X \setminus Y)\rho| : \rho \in A\}$.

Proof. By the definitions of $\alpha'(\Gamma)$ and $\alpha'_w(\Gamma)$, we have $\alpha'(\Gamma) \leq \alpha'_w(\Gamma)$. Suppose that there exists a weakly matching of Γ which is not strong. Then it contains two arcs (λ, δ) and (δ, η) where λ, δ, η are all distinct, which can be viewed in Figure 1.

Hence $\eta = \delta\mu$ and $\delta = \lambda\rho$ for some $\mu, \rho \in A \subseteq E_m$. Since all minimal idempotents of $Fix(X, Y)$ are left zero elements, we obtain that $\delta = \lambda\rho = \lambda$ which is a contradiction. Thus every weakly matching is also strong. This implies that $\alpha'_w(\Gamma) \leq \alpha'(\Gamma)$. Consequently, $\alpha'_w(\Gamma) = \alpha'(\Gamma)$.

Next, let $\mu \in A$ be such that $|(X \setminus Y)\mu| = q$, that is, $|(X \setminus Y)\mu| = \max\{|(X \setminus Y)\rho| : \rho \in A\}$. For convenience, let $X \setminus Y = \{x_1, x_2, \dots, x_r\}$ for some $r \in \mathbb{N}$. Furthermore, let $W = \{\eta \in E_m : (X \setminus Y)\eta \cap (X \setminus Y)\mu \neq \emptyset\}$. For each $\eta \in W$, we define $\lambda_\eta \in Fix(X, Y)$ as follows:

$$x\lambda_\eta = \begin{cases} x & \text{if } x \in Y; \\ x_i\eta & \text{if } x = x_i \in X \setminus Y \text{ and } x_i\eta \notin (X \setminus Y)\mu; \\ x_j & \text{if } x = x_i \in X \setminus Y \text{ and } x_i\eta = x_j\mu \end{cases}$$

where $j = \min\{t : x_t\mu = x_i\eta\}$.

By the definition of λ_η , it is obvious that $\lambda_\eta \notin E_m$. Moreover, we obtain that $\lambda_\eta\mu = \eta$ which yields that $(\lambda_\eta, \eta) \in E(\Gamma)$. If $\lambda_{\eta_1} = \lambda_{\eta_2}$, then $\eta_1 = \lambda_{\eta_1}\mu = \lambda_{\eta_2}\mu = \eta_2$. Therefore, $M = \{(\lambda_\eta, \eta) : \eta \in W\}$ is a matching of Γ . We conclude that

$$\alpha'(\Gamma) \geq |M| = |W| = \sum_{k=1}^q \binom{q}{k} \binom{r}{k} k!(m-k)^{r-k},$$

where $q = \max\{|(X \setminus Y)\rho| : \rho \in A\}$.

Next, we prove the upper bound of $\alpha'(\Gamma)$. By Lemma 3.17, we have $\alpha'(\Gamma) \leq |Fix(X, Y)| - \alpha(\Gamma)$. Further, we obtain by Theorem 3.2 that $\alpha(\Gamma) \geq \max\{|E_m|, |Fix(X, Y) \setminus E_m| + |M|\}$. Hence $\alpha'(\Gamma) \leq |Fix(X, Y)| - \max\{|E_m|, |Fix(X, Y) \setminus E_m| + |M|\} = |X|^{X|Y|} - \max\{|E_m|, |Fix(X, Y) \setminus E_m| + |M|\}$. Consequently, the assertion is completely proved.

From Theorem 3.2, we obtain the lower bound of $\alpha(\Gamma)$. Moreover, by Lemma 3.17 and Theorem 3.18, we get the upper bound of $\alpha(\Gamma)$ as presented in the following corollary.

Corollary 3.19. Let $A \subseteq E_m$ be a connection set of Γ . Then

$$\max\{|E_m|, |Fix(X, Y) \setminus E_m| + |M|\} \leq \alpha(\Gamma) \leq |X|^{X|Y|} - \sum_{k=1}^q \binom{q}{k} \binom{r}{k} k!(m-k)^{r-k},$$

where

$$M = \left\{ \beta \in E_m : (X \setminus Y)\beta \cap \left(\bigcup_{\lambda \in A} (X \setminus Y)\lambda \right) = \emptyset \right\}$$

and $q = \max\{|(X \setminus Y)\rho| : \rho \in A\}$.

3.4 Matching Parameters of Γ Related to Permutations

The last section shows results for matching parameters of Γ , whose connection sets are contained in H_{id_X} , including $\alpha'(\Gamma)$ and $\alpha'_w(\Gamma)$. To start the section, we need to prescribe some notations which are useful for our results. Let C_d denote a directed cycle with d vertices. Moreover, let λ be an element in a group H_{id_X} . The *order* $|\lambda|$ of λ is defined to be the smallest positive integer d such that $\lambda^d = id_X$. As the fact that, in this situation, $\Gamma[E_m]$ is exactly the union of isolated loops. Hence we will focus on $\Gamma[H_{id_X}]$ and $\Gamma[G]$, eventually.

Theorem 3.20. Let $A \subseteq H_{id_X}$ be a connection set of Γ and $|X \setminus Y| = r$.

(i) If A contains an even-order element, then

$$\alpha'(\Gamma[H_{id_X}]) = \left\lfloor \frac{r!}{2} \right\rfloor.$$

(ii) If A does not contain an even-order element, then

$$\alpha'(\Gamma[H_{id_X}]) = \left\lfloor \frac{k}{2} \right\rfloor \frac{r!}{k},$$

where $k = \max\{|\lambda| : \lambda \in A\}$.

Proof. (i) Assume that A contains an even-order element, say $\delta \in A$. That is $|\delta| = d$ for some even positive integer d . Consider the Cayley subdigraph $\text{Cay}(\langle \delta \rangle, \{\delta\})$ of Γ , we get that $\text{Cay}(\langle \delta \rangle, \{\delta\}) \cong C_d$. Moreover, we can simply observe that $\text{Cay}(H_{id_X}, \{\delta\})$ is the disjoint union of $\frac{|H_{id_X}|}{|\delta|}$ subdigraphs of Γ where each subdigraph is isomorphic to C_d . It is well known that $\alpha'(C_d) = \frac{d}{2}$. Therefore,

$$\alpha'(\Gamma[H_{id_X}]) = \frac{|H_{id_X}|}{|\delta|} \left(\frac{d}{2} \right) = \frac{r!}{2}.$$

(ii) Assume that A does not contain an even-order element. Let $\mu \in A$ be such that $|\mu| = k = \max\{|\lambda| : \lambda \in A\}$. Similarly to the proof of (i), we obtain that

$$\alpha'(\Gamma[H_{id_X}]) = \frac{|H_{id_X}|}{|\mu|} \left\lfloor \frac{k}{2} \right\rfloor = \frac{r!}{k} \left\lfloor \frac{k}{2} \right\rfloor.$$

Theorem 3.21. Let $A \subseteq H_{id_X}$ be a connection set of Γ and $|X \setminus Y| = r$. Then

$$\alpha'(\Gamma[G]) \leq |G/R| \left\lfloor \frac{|\langle A \rangle|}{2} \right\rfloor \left\lceil \frac{r!}{|\langle A \rangle|} \right\rceil.$$

Proof. By Lemma 3.14, we obtain that $\Gamma[G]$ is the disjoint union of $|G/R|$ subdigraphs which each of them is induced from an $\mathcal{R}$ -class of G . Let R be an $\mathcal{R}$ -class of G with maximum cardinality. Then the number of elements in R is exactly $\binom{r}{r-1}(r-1)!$. That is, $|V(\Gamma[R])| = r!$. Furthermore, we get from Lemma 2.3 that $\Gamma[G]$ is locally connected. Hence every vertex of $\Gamma[G]$ lies on a directed cycle. Moreover, we can observe that each directed cycle has length less than or equal to $|\langle A \rangle|$ and the matching number of the directed cycle is less than or equal to $\left\lfloor \frac{|\langle A \rangle|}{2} \right\rfloor$. Therefore, an upper bound of $\alpha'(\Gamma[G])$ is achieved.

Theorem 3.22. Let $A \subseteq H_{id_X}$ be a connection set of Γ and $|X \setminus Y| = r$.

$$(i) \alpha'_w(\Gamma[H_{id_X}]) = r!.$$

$$(ii) \alpha'_w(\Gamma[G]) \leq |G/R| |\langle A \rangle| \left\lceil \frac{r!}{|\langle A \rangle|} \right\rceil.$$

Proof. From $A \subseteq H_{id_X}$, we have by Lemma 2.3 that Γ is locally connected. Further, by Theorem 2.1, we know that Γ is the disjoint union of induced subdigraphs $\Gamma[E_m]$, $\Gamma[H_{id_X}]$ and $\Gamma[G]$. Hence $\Gamma[H_{id_X}]$ and $\Gamma[G]$ are also locally connected. This implies that every vertex, except isolated loops, lies on a directed cycle. Let $\delta \in A$. By the similar argument mentioned in the proof of Theorem 3.20(i), $\text{Cay}(H_{id_X}, \{\delta\})$ is the disjoint union of $\frac{|H_{id_X}|}{|\delta|}$ subdigraphs of Γ where each subdigraph is isomorphic to C_d where $d = |\delta|$. As the fact that $\alpha'_w(C_d) = d$, we conclude that

$$\alpha'_w(\Gamma[H_{id_X}]) = \left(\frac{|H_{id_X}|}{|\delta|} \right) d = |H_{id_X}| = r!.$$

(ii) By applying the proof of the upper bound for $\alpha'(\Gamma[G])$ in Theorem 3.21, we get that each directed cycle has length less than or equal to $|\langle A \rangle|$. Thus the weakly matching number of the directed cycle is less than or equal to $|\langle A \rangle|$. Therefore,

$$\alpha'_w(\Gamma[G]) \leq |G/R| |\langle A \rangle| \left\lceil \frac{r!}{|\langle A \rangle|} \right\rceil.$$

Finally, we propose some illustrative examples with graphical representations of small Cayley digraphs under different connection sets. The following examples introduce certain facts for improving the accessibility of the theoretical results. Let $X = \{1, 2, 3, 4\}$ and $Y = \{1, 2\}$. All 16 elements of $\text{Fix}(X, Y)$ can be expressed as follows:

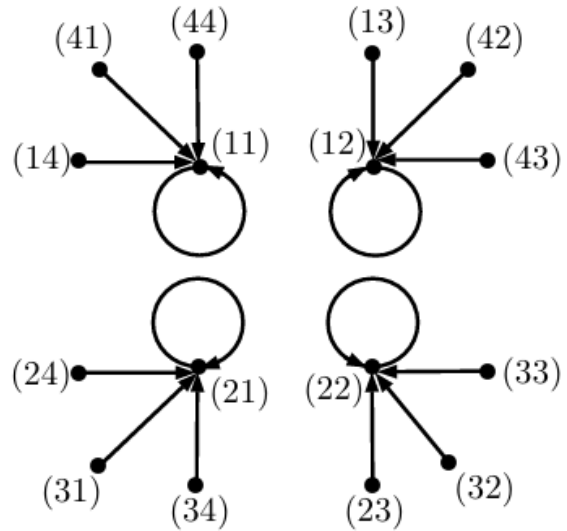


Figure 2. $\text{Cay}(\text{Fix}(X, Y), A)$ when $A = \{(21)\} \subseteq E_m$
Reprinted from (Nupo and Pookpienlert, 2020)

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & x_1 & x_2 \end{pmatrix} \quad \text{where } x_1, x_2 \in X.$$

Conveniently, the above expression will be shortly written as $(x_1 x_2)$.

Example 3.23. Let $A = \{(21)\} \subseteq E_m$. The Cayley digraph Γ of $\text{Fix}(X, Y)$ with respect to the connection set A is shown in Figure 2. We can observe that the set M in Theorem 3.2 is empty. Moreover, we can investigate that $\text{Fix}(X, Y) \setminus E_m$ is the maximum independent set of Γ which yields that $\alpha(\Gamma) = |\text{Fix}(X, Y)| - |E_m| = 16 - 4 = 12$. However, $\text{Fix}(X, Y)$ is a weakly independent set of Γ and thus $\alpha_w(\Gamma) = 16 = |\text{Fix}(X, Y)| = |X|^{|X|-|Y|}$. Furthermore, $\alpha_p(\Gamma) = 12 = \alpha(\Gamma)$ and $\alpha_{wp}(\Gamma) = 16 = \alpha_w(\Gamma)$. In addition, the set $\{(41), (11), (42), (12), (31), (21), (32), (22)\}$ is the maximum (weakly) matching of Γ . Therefore, $\alpha'(\Gamma) = 4 = \alpha'_w(\Gamma)$ which satisfies the bounds presented in Theorem 3.18.

Example 3.24. Let $A = \{(43)\} \subseteq H_{id_X}$. The Cayley digraph Γ of $\text{Fix}(X, Y)$ with respect to the connection set A is shown in Figure 3. It is not hard to compute that $\alpha(\Gamma[E_m]) = 4 = |E_m|$ and further, we have

$$\alpha(\Gamma[E_m]) = \alpha_w(\Gamma[E_m]) = \alpha_p(\Gamma[E_m]) = \alpha_{wp}(\Gamma[E_m]).$$

For the induced subdigraph $\Gamma[H_{id_X}]$, we obtain that $\alpha(\Gamma[H_{id_X}]) = 1$ which corresponds to the bounds stated in Theorem 3.9. We conclude that $\alpha(\Gamma[G]) = 5 = \alpha_w(\Gamma[G])$ with the maximum independent set $\{(14), (41), (24), (42), (33)\}$ of $\Gamma[G]$. These parameters satisfy the bounds mentioned in Theorem 3.15 where $|G/R| = 5$, $l = 0$, $|G| = 10$ and $A = A^{-1}$. Moreover, by the Cayley digraph Γ shown in Figure 3, we observe that $\alpha(\Gamma) = \alpha_w(\Gamma) = \alpha_p(\Gamma) = \alpha_{wp}(\Gamma) = 10$. In addition, we obtain that $\alpha'(\Gamma) = 6$ and $\alpha'_w(\Gamma) = 12$.

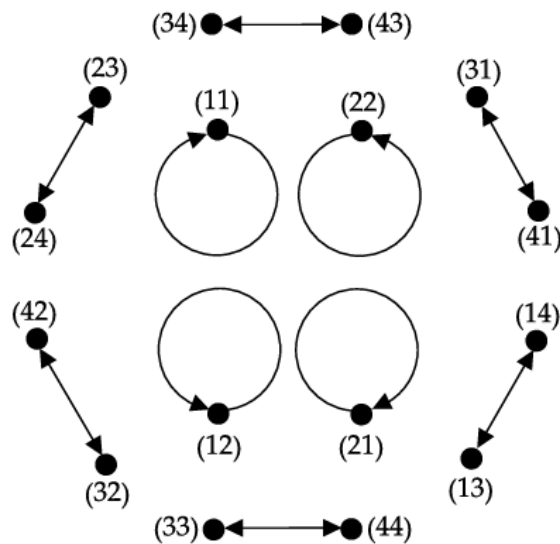


Figure 3. $\text{Cay}(\text{Fix}(X, Y), A)$ when $A = \{(43)\} \subseteq H_{id_X}$

4. CONCLUSIONS

In this paper, the independence and matching parameters have been studied for the Cayley digraph Γ of $Fix(X, Y)$. Our main results have been separated into two parts corresponding to minimal idempotents and permutations, respectively. We have found that $\alpha(\Gamma) = \alpha_p(\Gamma)$ and $\alpha_w(\Gamma) = \alpha_{wp}(\Gamma)$ whenever a connection set is contained in E_m . Moreover, we have obtained that $\alpha'(\Gamma) = \alpha'_w(\Gamma)$. For the case where a connection set is contained in H_{id_X} , we have proved that

$$\alpha(\Gamma[E_m]) = \alpha_w(\Gamma[E_m]) = \alpha_p(\Gamma[E_m]) = \alpha_{wp}(\Gamma[E_m]) = |E_m|.$$

Furthermore, we have provided the lower and upper bounds for $\alpha(\Gamma[H_{id_X}])$, $\alpha(\Gamma[G])$ and $\alpha_w(\Gamma[G])$. We have also investigated that $\alpha_p(\Gamma) = \alpha(\Gamma) = \alpha_w(\Gamma)$. In addition, some results of the matching parameters for induced subdigraphs of Γ relative to permutations have been presented. Finally, the explicit examples of Cayley digraphs of $Fix(X, Y)$ have been illustrated to improve the accessibility of the theoretical results.

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