

On the Spectral Characterization of Prime Coprime Graphs Associated with Dihedral Groups

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Abstract

Spectral graph theory and group theory help reveal how the structure of graphs shapes their mathematical properties. This research aims to analyze the spectral descriptors of the prime coprime graph of a dihedral group associated with the Laplacian, normalized Laplacian, and Seidel matrices. We also establish a method for determining the eigenvalues of these matrices. The results show that the Laplacian, signless Laplacian, and normalized Laplacian energies of the obtained graph are always even integers, whereas the adjacency and Seidel energies are never odd integers. These findings are consistent with previous studies and further strengthen the role of spectral descriptors in understanding algebraic graph structures.

Keywords

Prime Coprime Graph, Dihedral Group, Laplacian Matrix, Normalized Laplacian Matrix, Seidel Matrix

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1. INTRODUCTION

Spectral graph theory investigates graph structures through the eigenvalues and eigenvectors of matrices associated with graphs. The spectra of graph matrices, including the adjacency, Laplacian (Ding et al., 2024) and normalized Laplacian (Hu and Huo, 2026) matrices, encode fundamental information about connectivity, expansion, and other topological properties. In particular, graph eigenvalues, graph energy, spectral radius, and related energy decompositions have been extensively studied because of their close relationships with various structural invariants of graphs (Codogni and Lido, 2026; Rada, 2026).

Beyond its theoretical importance, spectral graph theory has become an influential framework for modeling complex systems across diverse scientific and engineering domains. Its applications include characterizing grain-boundary network evolution in anisotropic grain growth (Niño and Johnson, 2025), modeling magnetoencephalography-based frequency spectra in neuroscience (Verma et al. (2022)), and solving graph-based heat equations using Laplacian eigenvalues and eigenvectors (Cole et al., 2022). Spectral graph-based methods have also been extended to computational heat transfer modeling for

the efficient prediction of thermal histories in additive manufacturing (Cole et al., 2020). Moreover, closely related graph signal processing techniques have been applied to medical image analysis, where graph Laplacian spectral features extracted from MRI data support accurate and computationally efficient brain tumor classification (Dip and Meena, 2026). In addition, recent studies by Fujita et al. (2025, 2026), demonstrate the usefulness of specialized graph structures in representing complex relational systems, particularly in decision analytics and uncertainty-based frameworks.

Graphs associated with algebraic structures constitute an important research direction at the interface of graph theory and group theory. In this framework, a group can be represented as a graph whose vertices correspond to group elements, while adjacency is determined by prescribed algebraic relations. Several recent studies have examined algebraic structures using graph-theoretic and spectral approaches, including the construction of binary linear codes from zerodivisor graphs (Krisnawati et al., 2026) and the investigation of element orders in groups of higher even, odd, and prime orders (Mannan et al., 2022). Related work in algebra has further generalized commuting maps from rings to modules through the idealization construc-

tion (Fitriani et al., 2025). Various classes of group-based graphs have also been introduced, including non-commuting graphs (Salman et al., 2022), commuting graphs (Romdhini and Nawawi, 2026), and Cayley graphs (Hua et al., 2026), each capturing different structural aspects of the underlying group.

The graph was first introduced by Aschbacher (2000) under the term co-prime order graph, which was also used by (Sehgal et al., 2021). The same graph construction was subsequently published by (Adhikari and Banerjee, 2022) under the term prime coprime graph. Despite the different terminology, both terms refer to the same graph, in which two distinct elements of a finite group are adjacent whenever the greatest common divisor of their orders is either 1 or a prime number. Co-prime order graphs of finite groups have attracted increasing attention from both structural and spectral perspectives. Sehgal et al. (2021) investigated co-prime order graphs of finite Abelian and dihedral groups, including several of their graph-theoretic and spectral properties. Further structural investigations addressed connectivity and related properties Hao et al. (2022), positive genus Li et al. (2022), and divisor labeling Saini et al. (2024). More recently, degree-based matrix representations of co-prime order graphs have been studied to provide a broader spectral characterization beyond the adjacency matrix (Saini et al., 2025). These developments demonstrate that graph constructions based on element orders provide a useful framework for relating the algebraic structure of finite groups to structural and spectral graph invariants.

In a related direction, the construction and adjacency spectral properties of prime coprime graphs associated with dihedral groups have been investigated in Romdhini et al. (2025), while further structural properties of these graphs were discussed in Abdurahim and Romdhini (2026). Owing to their well-defined algebraic structure and rich interaction between rotational and reflectional elements, dihedral groups provide a natural framework for constructing and analyzing group-based graphs, including singular graphs with dihedral group action (Al-Tarimshawy and Siemons, 2021) and prime coprime graphs. Previous studies have established the adjacency spectrum (Romdhini et al., 2025), the Laplacian (Sehgal et al., 2021), and the normalized Laplacian spectrum (Saini et al., 2025) of the corresponding graph. However, the associated spectral energies and the Seidel spectral properties remain less explored. This represents a theoretical gap because different graph matrices provide complementary structural information that is not fully captured by the adjacency matrix alone. Motivated by this gap, the present study determines the Laplacian and normalized Laplacian energies from their corresponding spectral structures and derives the Seidel spectrum and energy of prime coprime graphs associated with dihedral groups. Thus, the main contribution of this work is to provide a broader multi-matrix spectral characterization and comparison of this class of graphs.

2. EXPERIMENTALL SECTION

Let Ω_G be the prime coprime graph of a group G . The concept of Ω_G naturally leads to the consideration of powers of prime numbers and products of two distinct primes. Its formal definition is presented as follows.

Definition 2.1 (Adhikari and Banerjee, 2022) The prime coprime graph of a finite group G with two distinct vertices u and v , are adjacent whenever the greatest common divisor of u and v is equal to 1 or a prime p , $(|u|, |v|) = 1$ or $(|u|, |v|) = p$.

Moreover, let $D(\Omega_G)$ denote the diagonal degree matrix of Ω_G and let $A(\Omega_G)$ denote its adjacency matrix. The following is the definition of Laplacian and signless Laplacian matrices of Ω_G .

Definition 2.2. (Brouwer and Haemers, 2011) The Laplacian matrix of Ω_G is given by

$$L(\Omega_G) = D(\Omega_G) - A(\Omega_G).$$

Moreover, the normalized matrix is a degree-normalized version of the Laplacian matrix, defined as follows.

Definition 2.3. (Brouwer and Haemers, 2011) The normalized Laplacian matrix of Ω_G is

$$N(\Omega_G) = I - D(\Omega_G)^{-\frac{1}{2}} A(\Omega_G) D(\Omega_G)^{-\frac{1}{2}}.$$

Definition 2.4. (Brouwer and Haemers, 2011) The signless Laplacian matrix of Ω_G is

$$Q(\Omega_G) = D(\Omega_G) + A(\Omega_G).$$

The Seidel matrix of a graph is defined based on the adjacency and non-adjacency relations between its vertices and is given as follows.

Definition 2.5. (Brouwer and Haemers, 2011) The Seidel matrix of Ω_G is

$$S(\Omega_G) = J - I - 2A(\Omega_G).$$

The characteristic polynomial of Laplacian matrix of $L(\Omega_G)$ is written by

$$P(L(\Omega_G)) = |\lambda I_n - L(\Omega_G)|.$$

The roots of $P(L(\Omega_G)) = 0$ are the eigenvalues of $L(\Omega_G)$, and denoted by $\lambda_1, \lambda_2, \dots, \lambda_n$. The spectral radius of $L(\Omega_G)$ is the largest absolute values of λ_i , for $i = 1, 2, \dots, n$ (Horn and Johnson, 1985). The energy of $L(\Omega_G)$ is

$$E(L(\Omega_G)) = \sum_{i=1}^n |\lambda_i|.$$

These terms also apply to normalized Laplacian and Seidel matrices (Gutman, 1978).

The dihedral group is one of the fundamental classes of finite groups in abstract algebra. It represents the group of symmetries of a regular polygon, consisting of rotations and

reflections. For a positive integer n , the dihedral group D_{2n} has order $2n$ and is commonly presented as

$$D_{2n} = \langle a, b \mid a^n = b^2 = e, bab = a^{-1} \rangle,$$

where a denotes a rotation, b denotes a reflection, and e is the identity element (Aschbacher, 2000). Its elements can be written as

$$D_{2n} = \{e, a, a^2, \dots, a^{n-1}, b, ab, a^2b, \dots, a^{n-1}b\}.$$

Accordingly, the prime coprime graph of D_{2n} is denoted by $\Omega_{D_{2n}}$, and the degree formula for the graph $\Omega_{D_{2n}}$ can be extended in a general form, as stated in the following theorem.

Theorem 2.6. (Abdurahim and Romdhini, 2026) For $n = p^k$ with p is prime and $k > 2$ is a natural number, then the degree of each vertex in $\Omega_{D_{2n}}$ is

1. $d(e) = 2p^k - 1$;
2. $d(a^i) = 2p^k - 1$, for $i = c \cdot p^{k-1}$ with $c \in \{1, 2, \dots, p-1\}$;
3. $d(a^i) = p^k + n$, for $i = c \cdot p^{k-1}$ with $c \in \{1, 2, \dots, p-1\}$;
4. $d(a^ib) = 2p^k - 1$, for $i \in \{0, 1, 2, \dots, p^k - 1\}$.

Furthermore, the following theorem provides a useful tool for simplifying the characteristic polynomial of a matrix, which will be employed in the next section.

Theorem 2.7. (Gantmacher, 1959) If a square matrix

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

can be partitioned into four blocks, where $|A| \neq 0$, then the determinant of M is

$$|M| = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = |A| |D - CA^{-1}B|.$$

3. RRESULT AND DISCUSSION

This section begins with the derivation of a general formula for the determinant of the resulting matrix.

Theorem 3.1. If an $n \times n$ matrix

$$M = \begin{pmatrix} a & bJ_{1 \times n_1} & cJ_{1 \times n_2} \\ bJ_{n_1 \times 1} & (d-e)I_{n_1} + eJ_{n_1} & bJ_{n_1 \times n_2} \\ cJ_{n_2 \times 1} & bJ_{n_2 \times n_1} & (a-c)I_{n_2} + cJ_{n_2} \end{pmatrix},$$

where $n = n_1 + n_2 + 1$, then characteristic polynomial of M is

Proof. The characteristic formula of M is the determinant of $\lambda I_n - M$ as follows.

$$P_M(\lambda) = \begin{vmatrix} \lambda - a & -bJ_{1 \times n_1} & -cJ_{1 \times n_2} \\ (a-b-\lambda)J_{n_1 \times 1} & (\lambda-d+e)I_{n_1} + (b+e)J_{n_1} & (c-b)J_{n_1 \times n_2} \\ (a-c-\lambda)J_{n_2 \times 1} & 0_{n_2 \times n_1} & (\lambda-a+c)I_{n_2} \end{vmatrix}$$

Let R_i and C_i represent the i -th row and column of the matrix, respectively. Assume that R'_i and C'_i denote the updated i -th row and column obtained after performing elementary

row and column transformations on R_i and C_i , respectively. These apply to $P_M(\lambda)$ as follows:

We replace the $n_1 + i$ -rows with the row operation:

$$R_{n_1+i} \rightarrow R_{n_1+i} - R_1,$$

for $i = 1, 2, \dots, n_1, n_1+1, n_1+2, \dots, n_1+n_2$. We can express $P_M(\lambda)$ as

We transform the first column by adding to it the columns $C_2, C_3, \dots, C_{n_1+1}$, or in other words,

$$C_2 \rightarrow C_2 + C_3 + C_4 + \dots + C_{n_1+1}.$$

Followed by replacing the first column and elements of $n_1 + 2, n_1 + 3, \dots, n_1 + n_2 + 1$, or

$$C_1 \rightarrow C_{n_1+2} + C_{n_1+3} + \dots + C_{n_1+n_2+1}.$$

Based on Theorem 2.7, we can write the characteristic polynomial M as follows.

$$P_M(\lambda) = (\lambda - d + e)^{n_1-1} (\lambda - a + c)^{n_2} [(\lambda - a - cn_2) ((\lambda - d + bn_1 + e(1 - n_1)) + bn_1(cn_2 - b(n_2 + 1) + a - \lambda))].$$

The following theorem gives the explicit form of the Laplacian, normalized Laplacian, and Seidel matrices of $\Omega_{D_{2n}}$. This matrix representation becomes the basis for deriving the characteristic polynomial, spectral radius, and energy of the graph.

Theorem 3.2. Let $\Omega_{D_{2n}}$ be the prime coprime graph on D_{2n} , where $n = p^k$ and integer $k \geq 2$. Then the Laplacian energy of $L(\Omega_{D_{2n}})$ is

$$E(L(\Omega_{D_{2n}})) = (p^k + p)(3p^k - p - 1).$$

Proof. We label the set G_1 as $\{a, a^2, \dots, a^{n-1}\}$ and G_2 as $\{b, ab, a^2b, \dots, a^{n-1}b\}$. Considering the definition of the prime coprime graph, every vertex in G_1 is always adjacent to each vertex of G_2 . Besides, every vertex in G_2 is also always adjacent to each vertex in that itself. Based on Definition 2.2 and Theorem 2.6, we see that $2n$ Laplacian matrix for $\Omega_{D_{2n}}$.

$$L(\Omega_{D_{2n}}) = \begin{pmatrix} 2p^k-1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & -1 \\ -1 & p^k+p & 0 & \dots & 0 & -1 & -1 & \dots & -1 \\ -1 & 0 & p^k+p & \dots & 0 & -1 & -1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & 0 & \dots & p^k+p & -1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & 2p^k-1 & -1 & \dots & -1 \\ -1 & -1 & -1 & \dots & -1 & -1 & 2p^k-1 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & -1 & -1 & -1 & \dots & 2p^k-1 \end{pmatrix}.$$

or in other words, we can write as the block matrices:

$$L(\Omega_{D_{2n}}) = \begin{pmatrix} 2p^k-1 & -J_{1 \times (p^k-1)} & -J_{1 \times p^k} \\ -J_{(p^k-1) \times 1} & (p^k+p)I_{p^k-1} & -J_{(p^k-1) \times p^k} \\ -J_{p^k \times 1} & -J_{p^k \times (p^k-1)} & (2p^k-1)I_{p^k} + J_{p^k} \end{pmatrix}.$$

By Theorem 3.1 where $a = 2p^k - 1$, $b = c = -1$, $d = p^k + p$, $e = 0$, $n_1 = p^k - p$, $n_2 = p^k - 1$, we have

$$\begin{aligned}
 P_{L(\Omega_{D_{2n}})}(\lambda) &= (\lambda - d + e)^{n_1-1} (\lambda - a + c)^{n_2} \\
 &\quad \times [(\lambda - a - cn_2)(\lambda - d + bn_1 + e(1 - n_1)) \\
 &\quad + bn_1(cn_2 - b(n_2 + 1) + a - \lambda)]. \\
 &= (\lambda - (p^k + p))^{p^k-p-1} (\lambda - (2p^k))^{p^k-1} \\
 &\quad \times \left[(\lambda - (p^k + p))(\lambda - (p^k - p) - (p^k - p) + 1) \right. \\
 &\quad \left. - (p^k - p) \left(-(p^k - p + 1) - (p^k - p + 1) + 2p^k - 1 - \lambda \right) \right]. \\
 &= (\lambda - p^k - p)^{p^k-p-1} (\lambda - 2p^k)^{p^k-1} \\
 &\quad \times \left[(\lambda - p^k + p)(\lambda - 2p^k + 1) - (p^k - p)(2p^k - \lambda) \right]. \\
 &= (\lambda - p^k - p)^{p^k-p-1} (\lambda - 2p^k)^{p^k-1} \\
 &\quad \times \left[(\lambda - p^k + p)(\lambda - 2p^k) \right].
 \end{aligned}$$

$$P_{L(\Omega_{D_{2n}})}(\lambda) = (\lambda - 2p^k)^{p^k} (\lambda - p^k - p)^{p^k-p}$$

Eigenvalues of $L(\Omega_{D_{2n}})$ are $\lambda_1 = 0$ of multiplicity 1, $\lambda_2 = 2p^k$ of multiplicity $p^k + p$, and $\lambda_3 = p^k + p$ of multiplicity $p^k - p - 1$. This Laplacian spectrum is consistent with Theorem 4.5 of (Sehgal et al., 2021), after adjusting the notation for the dihedral group parameters. While their result establishes the Laplacian spectrum, the present theorem uses this spectral structure to determine the corresponding Laplacian energy explicitly. The spectral radius of $L(\Omega_{D_{2n}})$ is $\rho(L(\Omega_{D_{2n}})) = 2p^k$, and the Laplacian energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned}
 E_L(\Omega_{D_{2n}}) &= |0| + (p^k + p)|2p^k| + (p^k - p - 1)|p^k + p| \\
 &= (p^k + p)(3p^k - p - 1).
 \end{aligned}$$

Having established the Laplacian energy in the previous theorem, we now determine the normalized Laplacian energy of $\Omega_{D_{2n}}$.

Theorem 3.3. Let $\Omega_{D_{2n}}$ be the prime coprime graph on D_{2n} , where $n = p^k$ and integer $k \geq 2$. Then the normalized Laplacian energy of $\Omega_{D_{2n}}$ is

$$E_N(\Omega_{D_{2n}})(\lambda) = 2p^k.$$

Proof. By the same argument as in Theorem 3.2, we can construct $N(\Omega_{D_{2n}})$ as follows.

$$N(\Omega_{D_{2n}}) = \begin{pmatrix} \frac{1}{2p^k-1} & -\frac{1}{\sqrt{(2p^k-1)(p^k+p)}} & -\frac{1}{\sqrt{(2p^k-1)(p^k+p)}} & \cdots & -\frac{1}{\sqrt{(2p^k-1)(2p^k-1)}} & \cdots \\ \frac{1}{\sqrt{(2p^k-1)(p^k+p)}} & \frac{p^k+p}{p^k+p} & 0 & \cdots & -\frac{1}{p^k+p} & \cdots \\ \frac{1}{\sqrt{(2p^k-1)(p^k+p)}} & 0 & \frac{p^k+p}{p^k+p} & \cdots & -\frac{1}{p^k+p} & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \\ -\frac{1}{\sqrt{(2p^k-1)(p^k+p)}} & 0 & 0 & \frac{p^k+p}{p^k+p} & \cdots & \cdots \\ -\frac{1}{2p^k-1} & -\frac{1}{p^k+p} & -\frac{1}{p^k+p} & \cdots & -\frac{1}{p^k+p} & 1 \cdots \end{pmatrix}$$

or in other words, we can write as the block matrices:

$$N(\Omega_{D_{2n}}) = \begin{pmatrix} \frac{1}{2p^k-1} & -\frac{1}{\sqrt{(2p^k-1)(p^k+p)}} J_{1 \times (p^k-p)} & -\frac{1}{2p^k-1} J_{1 \times p^k} \\ -\frac{1}{\sqrt{(2p^k-1)(p^k+p)}} J_{(p^k-p) \times 1} & I_{p^k-p} & -\frac{1}{p^k+p} J_{(p^k-p) \times p^k} \\ -\frac{1}{2p^k-1} J_{p^k \times 1} & -\frac{1}{p^k+p} J_{p^k \times (p^k-p)} & I_{p^k} \end{pmatrix}$$

By Theorem 3.1 where

$$a = 1, \quad b = -\frac{1}{\sqrt{(2p^k-1)(p^k+p)}}, \quad c = -\frac{1}{2p^k-1},$$

$$d = 1, \quad e = 0, \quad n_1 = p^k - p, \quad n_2 = p^k + p - 1,$$

consequently

$$\begin{aligned}
 P_{N(\Omega_{D_{2n}})}(\lambda) &= \frac{\lambda(\lambda-1)^{p^k-p-1} (2p^k-1-\lambda(2p^k-1))^{p^k+p-1}}{(2p^k-1)^{p^k+p}} \\
 &\quad \times (3p^k-p-1-\lambda(2p^k-1)).
 \end{aligned}$$

Eigenvalues of $N(\Omega_{D_{2n}})$ are

$$\lambda_1 = 0$$

of multiplicity 1,

$$\lambda_2 = 1$$

of multiplicity $p^k - p - 1$,

$$\lambda_3 = \frac{2p^k}{2p^k-1}$$

of multiplicity $p^k + p - 1$, and

$$\lambda_4 = \frac{3p^k-p-1}{2p^k-1}$$

of multiplicity 1.

The normalized Laplacian spectrum obtained above is closely related to **Corollary 5.3.** (Saini et al., 2025). Using the block parameters $p^k + p$ and $p^k - p$ in their general result (Theorem

5.3) yields the same eigenvalues and multiplicities as those derived here. This theorem further determines the corresponding normalized Laplacian energy explicitly.

The spectral radius of $N(\Omega_{D_{2n}})$ is

$$\rho(N(\Omega_{D_{2n}})) = \frac{3p^k - p - 1}{2p^k - 1},$$

and the normalized Laplacian energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned} E_N(\Omega_{D_{2n}}) &= (p^k - p - 1)|1| + (p^k + p - 1) \left| \frac{2p^k}{2p^k - 1} \right| \\ &\quad + \left| \frac{3p^k - p - 1}{2p^k - 1} \right| \\ &= 2p^k. \end{aligned}$$

Next, we determine the Seidel energy of $\Omega_{D_{2n}}$.

Theorem 3.4. Let $\Omega_{D_{2n}}$ be the prime coprime graph on D_{2n} , where $n = p^k$ and integer $k \geq 2$. Then the Seidel energy of $\Omega_{D_{2n}}$ is

$$E_S(\Omega_{D_{2n}}) = 2p^k - 2 + 2\sqrt{1 - p^2 - 2p^k + 2p^{2k}}.$$

Proof. The construction of Seidel matrix of $\Omega_{D_{2n}}$ is as follows.

$$S(\Omega_{D_{2n}}) = \begin{pmatrix} 0 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & -1 \\ -1 & 0 & 1 & \cdots & 1 & -1 & -1 & \cdots & -1 \\ -1 & 1 & 0 & \cdots & 1 & -1 & -1 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & 1 & 1 & \cdots & 0 & -1 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 & 0 & -1 & \cdots & -1 \\ -1 & -1 & -1 & \cdots & -1 & -1 & 0 & \cdots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \cdots & -1 & -1 & -1 & \cdots & 0 \end{pmatrix}.$$

or in other words, we can write as the block matrices:

$$S(\Omega_{D_{2n}}) = \begin{pmatrix} 0 & -J_{1 \times (p^k - p)} & -J_{1 \times (p^k + p - 1)} \\ -J_{(p^k - p) \times 1} & (J - I)_{(p^k - p)} & -J_{(p^k - p) \times (p^k + p - 1)} \\ -J_{(p^k + p - 1) \times 1} & -J_{(p^k + p - 1) \times (p^k - p)} & (J - I)_{(p^k + p - 1)} \end{pmatrix}.$$

By Theorem 3.1 where

$$a = 0, \quad b = c = -1, \quad d = 0, \quad e = 1,$$

$$n_1 = p^k - p, \quad n_2 = p^k + p - 1,$$

consequently

$$\begin{aligned} P_{S(\Omega_{D_{2n}})}(\lambda) &= (\lambda + 1)^{p^k - p - 1} (\lambda - 1)^{p^k + p - 1} \\ &\quad \times \left[(\lambda + p^k + p - 1)(\lambda - 2p^k + 2p + 1) \right. \\ &\quad \left. - (p^k - p)(1 - \lambda) \right]. \end{aligned}$$

Eigenvalues of $S(\Omega_{D_{2n}})$ are

$$\lambda_1 = -1$$

of multiplicity $p^k - p - 1$,

$$\lambda_2 = 1$$

of multiplicity $p^k + p - 1$, and

$$\lambda_3 = -p \pm \sqrt{1 - p^2 - 2p^k + 2p^{2k}}$$

of each multiplicity 1.

The spectral radius of $S(\Omega_{D_{2n}})$ is

$$\rho(S(\Omega_{D_{2n}})) = -p + \sqrt{1 - p^2 - 2p^k + 2p^{2k}},$$

and the Seidel energy of $\Omega_{D_{2n}}$ is

$$\begin{aligned} E_S(\Omega_{D_{2n}}) &= (p^k - p - 1)|-1| + (p^k + p - 1)|1| \\ &\quad + \left| -p + \sqrt{1 - p^2 - 2p^k + 2p^{2k}} \right| \\ &\quad + \left| -p - \sqrt{1 - p^2 - 2p^k + 2p^{2k}} \right| \\ &= 2p^k - 2 + 2\sqrt{1 - p^2 - 2p^k + 2p^{2k}}. \end{aligned}$$

We consider an example presented in (Romdhini et al., 2025). Let $\Omega_{D_{22}}$ denote the prime coprime graph of the dihedral group of order $2 \cdot 2^2$, as shown in Figure 1.

Example 3.5. The degree matrix of $\Omega_{D_{22}}$ is given below.

$$D(\Omega_{D_{22}}) = \begin{pmatrix} 7 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 7 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 7 \end{pmatrix}.$$

The adjacency matrix of $\Omega_{D_{22}}$ is given by

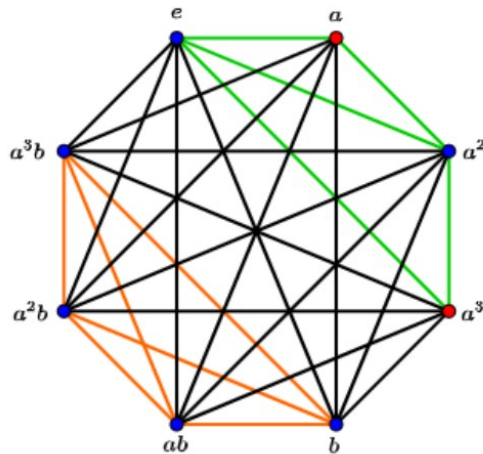


Figure 1. PRime Coprime Graph of D_{2^2}

$$A(\Omega_{D_{2^2}}) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \end{pmatrix}.$$

Recall that the Laplacian matrix is defined by

$$L(\Omega_{D_{2^2}}) = D(\Omega_{D_{2^2}}) - A(\Omega_{D_{2^2}}).$$

Hence, the Laplacian matrix of $\Omega_{D_{2^2}}$ is given by

$$L(\Omega_{D_{2^2}}) = \begin{pmatrix} 7 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 6 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 6 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 7 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 7 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 7 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 7 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 7 \end{pmatrix}.$$

The normalized matrix of $\Omega_{D_{2^2}}$ has the following form:

$$N(\Omega_{D_{2^2}}) = \begin{pmatrix} 1 & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{\sqrt{42}} & 1 & 0 & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} \\ -\frac{1}{\sqrt{42}} & 0 & 1 & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} \\ -\frac{1}{7} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 & -\frac{1}{7} \\ -\frac{1}{7} & -\frac{1}{\sqrt{42}} & -\frac{1}{\sqrt{42}} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & -\frac{1}{7} & 1 \end{pmatrix}.$$

The Seidel matrix of $\Omega_{D_{2^2}}$ can be written as follows.

$$S(\Omega_{D_{2^2}}) = \begin{pmatrix} 0 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 0 & 1 & -1 & -1 & -1 & -1 & -1 \\ -1 & 1 & 0 & -1 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & 0 & -1 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & 0 & -1 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & 0 & -1 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 & -1 \\ -1 & -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{pmatrix}.$$

The eigenvalues of the Laplacian matrix $L(\Omega_{D_{2^2}})$ are

$$\lambda_1 = 0, \quad \lambda_2 = 6, \quad \lambda_3 = \dots = \lambda_8 = 8.$$

The energy of the Laplacian matrix is

$$E(L(\Omega_{D_{2^2}})) = 54.$$

Meanwhile, the eigenvalues of the normalized Laplacian matrix $N(\Omega_{D_{2^2}})$ are

$$\lambda_1 = 0, \quad \lambda_2 = 1, \quad \lambda_3 = \frac{9}{7}, \quad \lambda_4 = \dots = \lambda_8 = \frac{8}{7}.$$

Hence, the energy of the normalized Laplacian matrix is

$$E(N(\Omega_{D_{2^2}})) = 8.$$

On the other hand, the eigenvalues of the Seidel matrix $S(\Omega_{D_{2^2}})$ are

$$\lambda_1 = -2 - \sqrt{21}, \quad \lambda_2 = -2 + \sqrt{21},$$

$$\lambda_3 = -1, \quad \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = 1.$$

Consequently, the energy of the Seidel matrix $S(\Omega_{D_{2^2}})$ is

$$E(S(\Omega_{D_{2^2}})) = 6 + 2\sqrt{21}.$$

Furthermore, the result on the adjacency energy of $\Omega_{D_{2^n}}$ from (Romdhini et al., 2025) was

$$E_A(\Omega_{D_{2n}}) = p^k + p - 1 + \sqrt{(p^k + p - 1)^2 + 4p^k(p^k - 1)}.$$

Meanwhile, the following result on the signless Laplacian energy of $\Omega_{D_{2n}}$ was obtained by (Adhikari and Banerjee, 2022).

$$\begin{aligned} E_Q(\Omega_{D_{2n}}) &= (p^k + p - 1)|2p^k| + (p^k - p - 1)|p^k + p| \\ &\quad + \left| 2p^k + p - 1 + \sqrt{2p^{2k} - p^k - p + 1} \right| \\ &\quad + \left| 2p^k + p - 1 - \sqrt{2p^{2k} - p^k - p + 1} \right| \\ &= (p^k + p)(3p^k - p - 1) \\ &= E_L(\Omega_{D_{2n}}). \end{aligned}$$

An interesting observation is that the Laplacian energy obtained in this paper always coincides with the signless Laplacian energy previously reported by Adhikari and Banerjee (2022). Although the Laplacian and signless Laplacian matrices have different spectral structures, the absolute deviations of their eigenvalues involved in the corresponding energy calculations reduce to the same closed-form expression,

$$(p^k + p)(3p^k - p - 1).$$

Thus, the equality

$$E_L(\Omega_{D_{2n}}) = E_Q(\Omega_{D_{2n}})$$

reflects an equality of the aggregate spectral energies rather than an equality of the two spectra.

For comparison purposes, Figure 2 shows the behavior of the adjacency, Laplacian, normalized Laplacian, and Seidel energies of $\Omega_{D_{2n}}$ with respect to p^k . As p^k increases, the energy values also increase, indicating a positive relationship between p^k and the corresponding energies.

As shown in Figure 2, the Laplacian and signless Laplacian energies of $\Omega_{D_{2n}}$ are greater than the adjacency, normalized Laplacian energy, and the Seidel energy. In addition, the adjacency, normalized Laplacian and Seidel energies show comparable trends, with values that are relatively close. This indicates that different matrix representations of the same graph may produce substantially different energy scales, reflecting the distinct structural information encoded by each matrix.

Another distinctive property occurs for the adjacency matrix, whose energy is consistently equal to twice its spectral radius (Romdhini et al., 2025). In contrast, this relationship does not hold for the Laplacian, signless Laplacian, normalized Laplacian, or Seidel matrices. This difference further illustrates that spectral energy depends strongly on the matrix representation used to describe the graph.

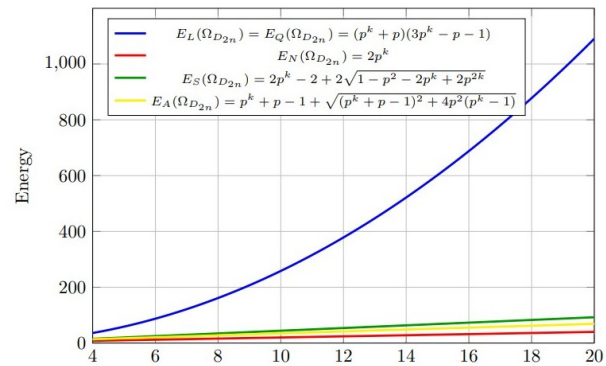


Figure 2. Comparison of the Adjacency, Laplacian, Signless Laplacian, Normalized, and Seidel Energies of $\Omega_{D_{2n}}$.

Moreover, the Laplacian, signless Laplacian and normalized Laplacian energies of $\Omega_{D_{2n}}$ are always an even integer. Meanwhile, the adjacency and Seidel energies of $\Omega_{D_{2n}}$ are never an odd integer. These results are consistent with those reported in previous studies (Bapat and Pati, 2004; Pirzada and Gutman, 2008).

4. CONCLUSIONS

This study highlights that the prime coprime graph of a dihedral group exhibits regular spectral behavior that can be interpreted through several degree- and 12 structure-based matrices. The derived spectral results broaden the understanding of how algebraic structure influences graph spectra and provide a useful framework for investigating related graph classes arising from finite groups. However, the present analysis is limited to prime coprime graphs associated with dihedral groups and to the matrix representations considered in this study. For future research, this work can be extended to prime coprime graphs of other finite groups, such as cyclic groups, quaternion groups, symmetric groups, and generalized dihedral groups. Further studies may also investigate other spectral descriptor.

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