

A Generalized Uncertainty Principle Involving Fractional Fourier Transform

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Abstract

It is well known that the fractional Fourier transform (FrFT) is a non-trivial generalization of the classical Fourier transform. This transform has been extensively utilized in signal analysis and image processing. The uncertainty principle is one of the fundamental results related to the FrFT. In the present paper, we first propose the different method in proving a variant of Heisenberg uncertainty principles associated with the FrFT. This method combines the Heisenberg-type uncertainty principle pertaining to the FrFT and the additive property of the FrFT. We then provide an alternative proof of the logarithmic inequality for the FRFT and find that this result differs slightly that in the literature. Finally, we explicitly expand the variant of Heisenberg uncertainty principle to a more general case.

Keywords

Fractional Fourier Transform, Heisenberg's Uncertainty Principle, Logarithmic Uncertainty Principle

Received: 14 June 2026, Accepted: 28 August 2026

<https://doi.org/10.26554/sti.2026.11.4.1679-1687>

1. INTRODUCTION

Just like the classical Fourier transform, which is an efficient method for solving partial differential equations, the fractional Fourier transform (FrFT) has been used to solve generalized partial differential equations (Mahor et al., 2021; Sulasteri et al., 2023). Currently, this transformation is the subject of the active ongoing research, which has attracted significant attention from a number of researchers due to its valuable application in various fields including signal processing, optics, quantum mechanics, telecommunications and image processing (see (Almeida, 1994; Ozaktas et al., 2001; Sharma and Josh, 2006; Zayed, 2018; Guanlei et al., 2009; Shah et al., 2021)).

On the flip side, the study on the uncertainty principle is one of the important problems in the Fourier transform (FT). In signal processing, an uncertainty principle relates a signal and its Fourier transform, which asserts that the signal cannot be localized in spatial and frequency domains. It means that the narrower a signal in the spatial domain, the wider signal is in the frequency domain and vice versa. In quantum physics, the uncertainty principle provides information about one cannot obtain simultaneously and precisely the values of position and momentum of a quantum article. Based on the brief facts, some researchers have attempted to formulate the uncertainty principle in various setting. For instance, in the study of many authors (see (Bahri, 2026; Bahri and Karim,

2023; Chen et al., 2025; Aloui and Brahim, 2023)), they delved into the uncertainty principles associated with the FrFT like Heisenberg's uncertainty principle, and logarithmic uncertainty principle. This paper further investigates the such uncertainty principles. Our contribution as shown Figure 1 is to provide an alternative proof of the uncertainty principles. We also propose a natural extension of the uncertainty principle to a more general case.

The highlights of the work are the followings:

- To propose a concise proof of a variant of Heisenberg uncertainty principle for the FrFT, compared with those found in the literature.
- To formulate and provide a new proof of logarithmic inequality associated with the considered FrFT.
- To extend the variant of Heisenberg uncertainty principle to a more general case and discuss some consequences of this principle.

The remainder of the present article is organized as follows. We present a brief review of the FrFT in Section 2 along with the relevant properties. The basic relation between the FrFT and the FT is also presented in this section. The formulation of various uncertainty principles for the FrFT are examined in detail in Section 3. This section also provides some examples to demonstrate the influence of the orders β and ϕ on the uncertainty principles for the FrFT, followed by the conclusion

in Section 4.

2. PRELIMINARIES

We take a quick look at the definition of the fractional Fourier transform (FrFT) and properties. It will be convenient to introduce a few notations, which will be important in this research.

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We denote by $L^p(\mathbb{R})$. The space of all complex-valued functions defined on $\mathbb{R}$ such that

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{-\infty}^{\infty} |f(t)|^p dt \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \quad (1)$$

and the space $L^2(\mathbb{R})$ is a Hilbert space with the inner product

$$\langle f, g \rangle = \int_{-\infty}^{\infty} f(t) \overline{g(t)} dt,$$

when $\overline{g(t)}$ is the complex conjugation of $g(t)$.

Next, we recall the definition and inverse of the FT and the FrFT as below.

Definition 1. Let the function f in $L^1(\mathbb{R})$, the Fourier transform (FT) for f is defined by (see (Debnath and Shah, 2015; Gröchenig, 2001; Bracewell, 2000))

$$\begin{aligned} \mathcal{F}\{f\}(\eta) &= \hat{f}(\eta) \\ &= \int_{-\infty}^{\infty} f(t) e^{-i\eta t} dt, \end{aligned} \quad (2)$$

for all $t \in \mathbb{R}$. If the function f and its FT $\hat{f}$ belongs to $L^1(\mathbb{R})$, the FT can be inverted using the formula

$$f(t) = \mathcal{F}^{-1}[\hat{f}(\eta)](t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(\eta) e^{i\eta t} d\eta, \quad \eta \in \mathbb{R}. \quad (3)$$

Definition 2. For every $f \in L^2(\mathbb{R})$, we define the FrFT for signal f with order φ as

$$\mathcal{F}^\varphi\{f\}(\eta) = \hat{f}^\varphi(\eta) = \int_{-\infty}^{\infty} f(t) K^\varphi(t, \eta) dt. \quad (4)$$

Here, the kernel $K^\varphi(t, \eta)$ is

$$K^\varphi(t, \eta) = \begin{cases} B_\varphi e^{i(t^2 + \eta^2) \frac{\cot \varphi}{2} - i\eta t \csc \varphi} & \varphi \in \mathbb{R} \setminus n\pi \\ \delta(t - \eta) & \varphi = 2n\pi \\ \delta(t + \eta) & \varphi = (2n \pm 1)\pi, \end{cases} \quad (5)$$

with $n \in \mathbb{Z}$ and

$$B_\varphi = (2\pi i \sin \varphi)^{-\frac{1}{2}} e^{\frac{i\varphi}{2}} = \sqrt{\frac{1 - i \cot \varphi}{2\pi}}. \quad (6)$$

The original signal f in terms of the FrFT defined by Equation (4) is evaluated using the following formula.

Definition 3. For every signal $f \in L^1(\mathbb{R})$ and its FrFT $\hat{f}^\varphi \in L^1(\mathbb{R})$, the inversion formula of the FrFT is expressed by

$$f(t) = \int_{-\infty}^{\infty} \hat{f}^\varphi(\eta) \overline{K^\varphi(t, \eta)} d\eta. \quad (7)$$

Here,

$$\overline{K^\varphi(t, \eta)} = \overline{B_\varphi} e^{-i(t^2 + \eta^2) \frac{\cot \varphi}{2} + i\eta t \csc \varphi}, \quad (8)$$

and

$$\overline{B_\varphi} = (-2\pi i \sin \varphi)^{-\frac{1}{2}} e^{-\frac{i\varphi}{2}}. \quad (9)$$

It should be observed that when $\varphi = \frac{\pi}{2}$, the FrFT definition described by (4) coincides with the FT definition in Equation (2), that is,

$$\mathcal{F}^{\frac{\pi}{2}}\{f\}(\eta) = \frac{1}{\sqrt{2\pi}} \mathcal{F}\{f\}(\eta) \quad (10)$$

By virtue of Definition 2, one can obtain the natural connection between the FrFT and FT in the form

$$\mathcal{F}^\varphi\{f\}(\eta) = B_\varphi e^{i\eta^2 \frac{\cot \varphi}{2}} \mathcal{F}\{h_\varphi\}(\eta \csc \varphi), \quad (11)$$

where

$$h_\varphi(t) = f(t) e^{it^2 \frac{\cot \varphi}{2}}. \quad (12)$$

Equations (11) and (12) are very helpful in obtaining some uncertainty inequalities related to the FrFT. With the help of identity (11) above, one also can derive Parseval's formula for the FrFT (see (Bahri and Karim, 2023; Li and Xu, 2012)), that is,

$$\langle f, g \rangle = \langle \mathcal{F}^\varphi\{f\}, \mathcal{F}^\varphi\{g\} \rangle, \quad (13)$$

where $f, g \in L^2(\mathbb{R})$. In particular,

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(t)|^2 dt = \int_{-\infty}^{\infty} |\mathcal{F}^\varphi\{f\}(\eta)|^2 d\eta = \|\mathcal{F}^\varphi\{f\}\|_2^2. \quad (14)$$

An useful property of the FrFT is the additive rule, namely,

$$\mathcal{F}^\varphi[\mathcal{F}^\alpha\{f\}] = \mathcal{F}^{\varphi+\alpha}\{f\}, \quad \forall f \in L^2(\mathbb{R}). \quad (15)$$

3. RESULTS AND DISCUSSIONS

This study explores three uncertainty principles related to the FrFT. First, we establish a variant of Heisenberg uncertainty principle whose proof combines the Heisenberg inequality and the additive property associated with the FrFT. Second, we provide the alternative proof of the logarithmic inequality. Third, we extend the variant of Heisenberg uncertainty inequality to a more general case.

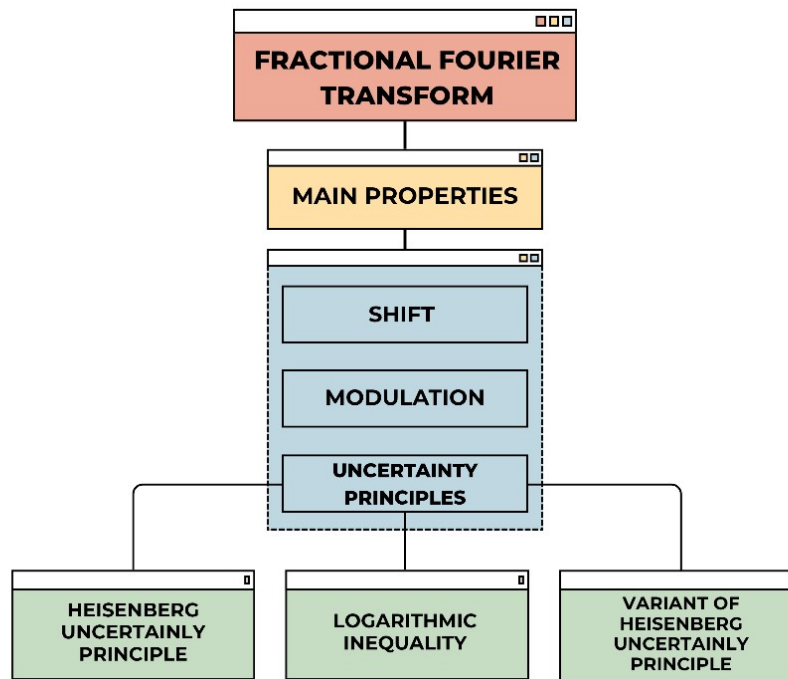


Figure 1. Research Flowchart for Fractional Fourier Transform

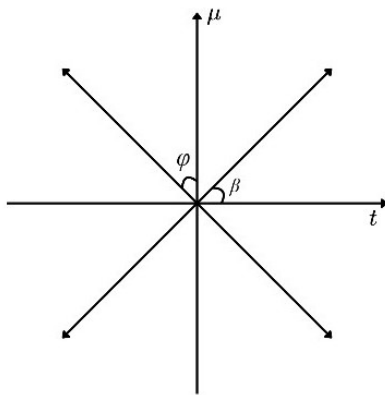


Figure 2. Parameters β , φ in FrFT Domain

3.1 Uncertainty Principles for FrFT

One of the most important results of the harmonic analysis is the uncertainty principle. It has played an important role in image processing and signal analysis. In this part, we provide an alternative proof of the uncertainty inequality for the FrFT by leveraging Heisenberg-type inequality pertaining to this transformation.

Theorem 1. For all orders α , φ such that the function $f \in L^2(\mathbb{R})$

and $\mathcal{F}^{\varphi+\alpha}\{f\} \in L^2(\mathbb{R})$, then the following inequality holds right:

$$\int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^{\alpha}\{f\}(\eta)|^2 d\eta \int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\alpha+\varphi}\{f\}(v)|^2 dv \geq \frac{|\sin \varphi|^2}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2 \quad (16)$$

If we set $\alpha + \varphi = \beta$ (see Figure 2), Equation (16) above changes to

$$\int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^{\alpha}\{f\}(\eta)|^2 d\eta \int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\beta}\{f\}(v)|^2 dv \geq \frac{|\sin(\beta - \alpha)|^2}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2 \quad (17)$$

This result is the same as that in Guanlei et al. (2009).

Proof. In view of the uncertainty principle for the FrFT (see (Shi et al., 2012)), we have

$$\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^{\varphi}\{f\}(\eta)|^2 d\eta \geq \frac{|\sin \varphi|^2}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2 \quad (18)$$

Since $f \in L^2(\mathbb{R})$ and $\mathcal{F}^{\alpha}\{f\} \in L^2(\mathbb{R})$, then we may replace

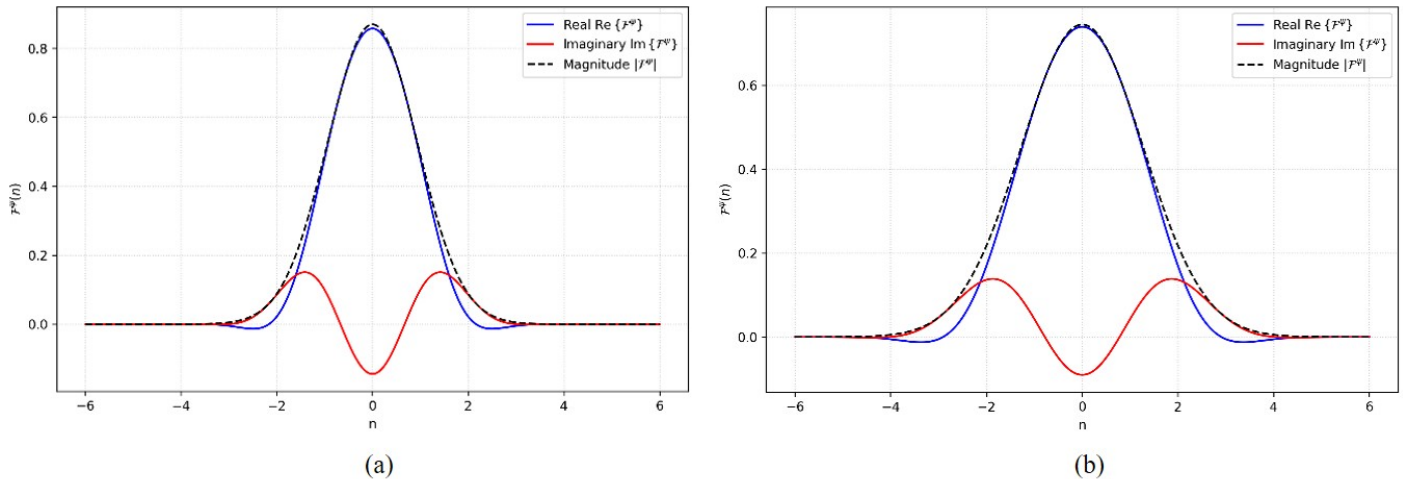


Figure 3. (a) The FrFT of Example 1 for $\varphi = \frac{\pi}{6}$, (b) The FrFT of Example 1 for $\varphi = \frac{\pi}{3}$

$f(t)$ by $\mathcal{F}^\alpha\{f\}(\eta)$ on both sides of Equation (18) to get

$$\int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^\alpha\{f\}(\eta)|^2 d\eta \int_{-\infty}^{\infty} v^2 |\mathcal{F}^\varphi\{\mathcal{F}^\alpha\{f\}\}(v)|^2 dv \geq \frac{|\sin \varphi|^2}{4} \left(\int_{-\infty}^{\infty} |\mathcal{F}^\alpha\{f\}(\eta)|^2 d\eta \right)^2 \quad (19)$$

With the aid of relations (14) and (15), the above expression may be rewritten in the form

$$\int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^\alpha\{f\}(\eta)|^2 d\eta \int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\alpha+\varphi}\{f\}(v)|^2 dv \geq \frac{|\sin \varphi|^2}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2. \quad (20)$$

Thus Equation (17) follows.

Some observations regarding Theorem 1 are summarized in the following remark.

Remark 1.

- Guanlei et al. (2009) obtained the uncertainty inequality of Theorem 1 by exploiting the basic relation between the FT and the FrFT.
- The proof of Theorem 1 above is carried out by leveraging the Heisenberg-type uncertainty principle pertaining to the FrFT and the additive rule property of the FrFT, whereas in the literature its proof uses the basic connection between the FT and the FRFT. Our technique is simpler than the method proposed by Guanlei et al. (2009).
- It should be noticed that for $\alpha = \frac{\pi}{2}$ and $\beta = 2\pi$, Equation (17) corresponds to

$$\int_{-\infty}^{\infty} t^2 |f(t)|^2 dt \int_{-\infty}^{\infty} \eta^2 |\mathcal{F}\{f\}(\eta)|^2 d\eta \geq \frac{\pi}{2} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2 \quad (21)$$

In Guanlei et al. (2009), the authors have proposed the logarithmic uncertainty principle associated with the FrFT in the following form:

$$\int_{-\infty}^{\infty} \ln |v| |\mathcal{F}^\varphi\{f\}(v)|^2 dv + \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}^\beta\{f\}(\eta)|^2 d\eta \geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \ln |\sin(\varphi - \beta)| \right) \int_{-\infty}^{\infty} |f(t)|^2 dt \quad (22)$$

The alternative form of the uncertainty principle (22) and its proof are presented in the next result.

Theorem 2. For every $f \in S(\mathbb{R})$ the following inequality holds

$$\int_{-\infty}^{\infty} \ln |v| |\mathcal{F}^\varphi\{f\}(v)|^2 dv + 2\pi \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}^\beta\{f\}(\eta)|^2 d\eta \geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \ln |\csc(\beta - \varphi)| \right) \int_{-\infty}^{\infty} |f(t)|^2 dt. \quad (23)$$

In this case, $S(\mathbb{R})$ is the Schwartz space on $\mathbb{R}$ and $\Gamma(\cdot)$ is usual gamma function.

Proof. Based on the logarithmic uncertainty principle for the FT, we have

$$\int_{-\infty}^{\infty} \ln |t| |f(t)|^2 dt + \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}\{f\}(\eta)|^2 d\eta \geq \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} \int_{-\infty}^{\infty} |f(t)|^2 dt. \quad (24)$$

Replacing $f(t)$ by $h_\varphi(t)$ defined by (12) and then taking $\eta \rightarrow \eta \csc \varphi$ in Equation (24), we conclude that

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |t| |h_{\varphi}(t)|^2 dt + \int_{-\infty}^{\infty} \ln |\eta \csc \varphi| |\mathcal{F}\{h_{\varphi}\}(\eta \csc \varphi)|^2 |\csc \varphi| d\eta \\ \geq \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} \int_{-\infty}^{\infty} |h_{\varphi}(t)|^2 dt. \end{aligned} \quad (25)$$

Therefore,

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |t| |h_{\varphi}(t)|^2 dt + \int_{-\infty}^{\infty} \ln |\eta \csc \varphi| \left| \frac{1}{|B_{\varphi}|} \left| e^{-i\eta^2 \frac{\cot \varphi}{2}} \mathcal{F}^{\varphi}\{f\}(\eta) \right|^2 \right. \\ \left. \times |\csc \varphi| d\eta \geq \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} \int_{-\infty}^{\infty} |h_{\varphi}(t)|^2 dt. \end{aligned} \quad (26)$$

This equation is the same as

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |t| \left| f(t) e^{it^2 \frac{\cot \varphi}{2}} \right|^2 dt + 2\pi |\sin \pi| \\ + \int_{-\infty}^{\infty} \ln |\eta \csc \varphi| \left| \frac{1}{|B_{\varphi}|} \left| e^{-i\eta^2 \frac{\cot \varphi}{2}} \mathcal{F}^{\varphi}\{f\}(\eta) \right|^2 \right| |\csc \varphi| d\eta \\ \geq \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} \int_{-\infty}^{\infty} \left| f(t) e^{it^2 \frac{\cot \varphi}{2}} \right|^2 dt. \end{aligned} \quad (27)$$

Simplifying Equation (27) will lead to

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |t| |f(t)|^2 dt + 2\pi \int_{-\infty}^{\infty} (\ln |\csc \varphi| + \ln |\eta|) |\mathcal{F}^{\varphi}\{f\}(\eta)|^2 d\eta \\ \geq \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} \int_{-\infty}^{\infty} |f(t)|^2 dt \end{aligned} \quad (28)$$

This equation implies that

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |t| |f(t)|^2 dt + 2\pi \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}^{\varphi}\{f\}(\eta)|^2 d\eta \\ \geq \frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} \int_{-\infty}^{\infty} |f(t)|^2 dt \\ - 2\pi \int_{-\infty}^{\infty} \ln |\csc \varphi| |\mathcal{F}^{\varphi}\{f\}(\eta)|^2 d\eta \end{aligned} \quad (29)$$

The use of Plancherel's identity (14) results in

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |t| |f(t)|^2 dt + 2\pi \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}^{\varphi}\{f\}(\eta)|^2 d\eta \\ \geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - 2\pi \ln |\csc \varphi| \right) \int_{-\infty}^{\infty} |f(t)|^2 dt \end{aligned} \quad (30)$$

Now, from Equation (30), we find that

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |v| |\mathcal{F}^{\alpha}\{f\}(v)|^2 dv + 2\pi \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}^{\alpha+\varphi}\{f\}(\eta)|^2 d\eta \\ \geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - 2\pi \ln |\csc \varphi| \right) \int_{-\infty}^{\infty} |\mathcal{F}^{\alpha}\{f\}(v)|^2 dv \end{aligned} \quad (31)$$

This equation simplifies to

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |v| |\mathcal{F}^{\alpha}\{f\}(v)|^2 dv + 2\pi \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}^{\alpha+\varphi}\{f\}(\eta)|^2 d\eta \\ \geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - 2\pi \ln |\csc \varphi| \right) \int_{-\infty}^{\infty} |f(t)|^2 dt \end{aligned} \quad (32)$$

By putting $\alpha + \varphi = \beta$ in Equation (32), we arrive at

$$\begin{aligned} \int_{-\infty}^{\infty} \ln |v| |\mathcal{F}^{\alpha}\{f\}(v)|^2 dv + 2\pi \int_{-\infty}^{\infty} \ln |\eta| |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta \\ \geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - 2\pi \ln |\csc(\beta - \alpha)| \right) \int_{-\infty}^{\infty} |f(t)|^2 dt, \end{aligned}$$

as desired.

We formulate a new generalization of Theorem 1, which is the main result of this present work. We will see that the uncertainty principle in Equation (16) is a special case of this generalized principle.

Theorem 3. Under the same situations as in Theorem 1, it holds:

$$\begin{aligned} \left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{\gamma}{\mu+\gamma}} \left(\int_{-\infty}^{\infty} \eta^{2\gamma} |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta \right)^{\frac{\mu}{\mu+\gamma}} \\ \geq \left(\frac{|\sin(\beta - \varphi)|^2}{4} \right)^{\frac{\mu\gamma}{\mu+\gamma}} \|f\|_2^2, \end{aligned} \quad (33)$$

where $\frac{1}{\mu} + \frac{1}{\mu'} = 1$, $\frac{1}{\gamma} + \frac{1}{\gamma'} = 1$ and $\mu, v \geq 1$.

Proof. By virtue of Hölder's inequality, we see that

$$\begin{aligned} \int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \\ = \int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\varphi}\{f\}(v)|^{\frac{2}{\mu}} |\mathcal{F}^{\varphi}\{f\}(v)|^{2-\frac{2}{\mu}} dv \\ \leq \left(\int_{-\infty}^{\infty} v^2 \left(|\mathcal{F}^{\varphi}\{f\}(v)|^{\frac{2}{\mu}} \right)^{\mu} dv \right)^{\frac{1}{\mu}} \\ \times \left(\int_{-\infty}^{\infty} \left(|\mathcal{F}^{\varphi}\{f\}(v)|^{2-\frac{2}{\mu}} \right)^{\frac{\mu}{\mu-1}} dv \right)^{1-\frac{1}{\mu}} \\ = \left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{1}{\mu}} \\ \times \left(\int_{-\infty}^{\infty} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{1-\frac{1}{\mu}} \\ = \left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{1}{\mu}} \\ \times \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^{1-\frac{1}{\mu}}, \end{aligned} \quad (34)$$

where in the last line we used Parseval's formula for the FrFT (14). Further, Equation (34) may be expressed in the form

$$\left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{1}{\mu}} \geq \frac{\int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv}{\|f\|_2^{\frac{2(\mu-1)}{\mu}}} \quad (35)$$

In a similar way, it is not difficult to verify that

$$\left(\int_{-\infty}^{\infty} \eta^{2\gamma} |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta \right)^{\frac{1}{\gamma}} \geq \frac{\int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta}{\|f\|_2^{\frac{2(\gamma-1)}{\gamma}}} \quad (36)$$

Combining relations (35) and (36) produces

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{1}{\mu}} \left(\int_{-\infty}^{\infty} \eta^{2\gamma} |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta \right)^{\frac{1}{\gamma}} \\ & \geq \frac{\int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta}{\|f\|_2^{2\left(\frac{\mu-1}{\mu} + \frac{\gamma-1}{\gamma}\right)}} \end{aligned} \quad (37)$$

By Equation (17), one gets

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{1}{\mu}} \left(\int_{-\infty}^{\infty} \eta^{2\gamma} |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta \right)^{\frac{1}{\gamma}} \\ & \geq \frac{|\sin(\beta - \varphi)|^2}{4} \frac{\|f\|_2^4}{\|f\|_2^{2\left(2 - \frac{\mu+\gamma}{\mu\gamma}\right)}} \end{aligned} \quad (38)$$

This equation may be simplified to

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{1}{\mu}} \left(\int_{-\infty}^{\infty} \eta^{2\gamma} |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta \right)^{\frac{1}{\gamma}} \\ & \geq \frac{|\sin(\beta - \varphi)|^2}{4} \|f\|_2^{2\left(\frac{\mu+\gamma}{\mu\gamma}\right)} \end{aligned} \quad (39)$$

Hence,

$$\begin{aligned} & \left(\int_{-\infty}^{\infty} v^{2\mu} |\mathcal{F}^{\varphi}\{f\}(v)|^2 dv \right)^{\frac{\gamma}{\mu+\gamma}} \left(\int_{-\infty}^{\infty} \eta^{2\gamma} |\mathcal{F}^{\beta}\{f\}(\eta)|^2 d\eta \right)^{\frac{\mu}{\mu+\gamma}} \\ & \geq \left(\frac{|\sin(\beta - \varphi)|^2}{4} \right)^{2\left(\frac{\mu\gamma}{\mu+\gamma}\right)} \|f\|_2^2 \end{aligned}$$

This is required result.

The uncertainty principles mentioned above are compared to existing literature shown in Table 1.

Remark 2.

It should be membered that for $\mu = \gamma = 1$ in Equation (33), we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta^2 |\mathcal{F}^{\alpha}\{f\}(\eta)|^2 d\eta \int_{-\infty}^{\infty} v^2 |\mathcal{F}^{\beta}\{f\}(v)|^2 dv \\ & \geq \frac{|\sin(\beta - \alpha)|^2}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2 \end{aligned}$$

3.2 Illustrative Examples

In this part, we provide illustrative examples to demonstrate the effect of orders β and ϕ on the FrFT uncertainty principles.

Example 1. Consider a Gaussian function defined over $\mathbb{R}$:

$$f(t) = e^{-\alpha t^2}, \quad \alpha > 0. \quad (40)$$

A direct computation shows that

$$\begin{aligned} \mathcal{F}^{\varphi}\{e^{-\alpha t^2}\}(\eta) &= B_{\varphi} \int_{-\infty}^{\infty} e^{-\alpha t^2} e^{i(t^2 + \eta^2)\frac{\cot \varphi}{2} - i t \eta \csc \varphi} dt \\ &= B_{\varphi} \int_{-\infty}^{\infty} e^{-(\alpha - \frac{i}{2} \cot \varphi)t^2 - i \eta \csc \varphi t + i \frac{\eta^2}{2} \cot \varphi} dt \end{aligned} \quad (41)$$

Using the fact (see (Tao et al., 2010)).

$$\int_{-\infty}^{\infty} e^{-At^2 \pm 2Bt + C} dt = \sqrt{\frac{\pi}{A}} e^{\frac{B^2}{A} + C}, \quad (42)$$

where $A, B, C \in \mathbb{C}$ with $A \neq 0$ and $\text{Re}(A) \geq 0$.

Thus Equation (41) becomes

$$\begin{aligned} \mathcal{F}^{\varphi}\{e^{-\alpha t^2}\}(\eta) &= \sqrt{\frac{1 - i \cot \varphi}{2\pi}} \sqrt{\frac{\pi}{\alpha - \frac{i}{2} \cot \varphi}} e^{-\frac{\eta^2 \csc^2 \varphi}{4\alpha - 2i \cot \varphi} + i \frac{\eta^2}{2} \cot \varphi} \\ &= \sqrt{\frac{1 - i \cot \varphi}{2\alpha - i \cot \varphi}} e^{-\frac{\eta^2 \csc^2 \varphi}{4\alpha - 2i \cot \varphi} + i \frac{\eta^2}{2} \cot \varphi} \end{aligned} \quad (43)$$

The Graph of the FrFT of Gaussian function at various angles is shown in Figure 3.

From Equation (43) we obtain

a) for $\varphi = \frac{\pi}{2}$

$$\begin{aligned} \mathcal{F}^{\frac{\pi}{2}}\{e^{-\alpha t^2}\}(\eta) &= \sqrt{\frac{1 - i \cot \frac{\pi}{2}}{2\alpha - i \cot \frac{\pi}{2}}} e^{-\frac{\eta^2 \csc^2 \frac{\pi}{2}}{4\alpha - 2i \cot \frac{\pi}{2}} + i \frac{\eta^2}{2} \cot \frac{\pi}{2}} \\ &= \sqrt{\frac{1}{2\alpha}} e^{-\frac{\eta^2}{4\alpha}} \end{aligned} \quad (44)$$

Applying (10) to (44) yields

$$\mathcal{F}\{e^{-\alpha t^2}\}(\eta) = \sqrt{2\pi} \mathcal{F}^{\frac{\pi}{2}}\{e^{-\alpha t^2}\}(\eta) = \sqrt{2\pi} \sqrt{\frac{1}{2\alpha}} e^{-\frac{\eta^2}{4\alpha}} = \sqrt{\frac{\pi}{\alpha}} e^{-\frac{\eta^2}{4\alpha}} \quad (45)$$

b) For $\varphi = \frac{\pi}{4}$

$$\begin{aligned} \mathcal{F}^{\frac{\pi}{4}}\{e^{-\alpha t^2}\}(\eta) &= \sqrt{\frac{1 - i \cot \frac{\pi}{4}}{2\alpha - i \cot \frac{\pi}{4}}} e^{-\frac{\eta^2 \csc^2 \frac{\pi}{4}}{4\alpha - 2i \cot \frac{\pi}{4}} + i \frac{\eta^2}{2} \cot \frac{\pi}{4}} \\ &= \sqrt{\frac{1 - i}{2\alpha - i}} e^{-\frac{2\eta^2}{4\alpha - 2i} + i \frac{\eta^2}{2}} \end{aligned} \quad (46)$$

Equation (45) shows that the Gaussian function in the FT domain is a special case of Gaussian function in the FrFT domain.

Table 1. A Comparison of Uncertainty Principles for FrFT

Uncertainty Principles by Guanlei et al. (2009)	Uncertainty Principles in the Present Work
$\int_{-\infty}^{\infty} \ln v \mathcal{F}^{\varphi}\{f\}(v) ^2 dv + \int_{-\infty}^{\infty} \ln \eta \mathcal{F}^{\beta}\{f\}(\eta) ^2 d\eta$ $\geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \ln \sin(\varphi - \beta) \right) \int_{-\infty}^{\infty} f(t) ^2 dt$	$\int_{-\infty}^{\infty} \ln v \mathcal{F}^{\varphi}\{f\}(v) ^2 dv$ $+ 2\pi \int_{-\infty}^{\infty} \ln \eta \mathcal{F}^{\beta}\{f\}(\eta) ^2 d\eta$ $\geq \left(\frac{\Gamma'(\frac{1}{4})}{\Gamma(\frac{1}{4})} - \ln \csc(\beta - \varphi) \right)$ $\times \int_{-\infty}^{\infty} f(t) ^2 dt$
$\left(\int_{-\infty}^{\infty} v^2 \mathcal{F}^{\varphi}\{f\}(v) ^2 dv \right) \left(\int_{-\infty}^{\infty} \eta^2 \mathcal{F}^{\beta}\{f\}(\eta) ^2 d\eta \right)$ $\geq \left(\frac{ \sin(\beta - \varphi) ^2}{4} \right) \ f\ _2^2,$	$\left(\int_{-\infty}^{\infty} v^{2\mu} \mathcal{F}^{\varphi}\{f\}(v) ^2 dv \right)^{\frac{\gamma}{\mu+\gamma}}$ $\left(\int_{-\infty}^{\infty} \eta^{2\gamma} \mathcal{F}^{\beta}\{f\}(\eta) ^2 d\eta \right)^{\frac{\mu}{\mu+\gamma}}$ $\geq \left(\frac{ \sin(\beta - \varphi) ^2}{4} \right)^{\frac{\mu\gamma}{\mu+\gamma}} \ f\ _2^2,$

Further, based on fact

$$\int_{-\infty}^{\infty} t^2 e^{-\alpha t^2} dt = \frac{\sqrt{\pi}}{2} \alpha^{-\frac{3}{2}}, \quad \alpha > 0, \quad (47)$$

then we obtain

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^{\varphi}\{e^{-\alpha t^2}\}(\eta) \right|^2 d\eta \\ &= \int_{-\infty}^{\infty} \eta^2 \left| \sqrt{\frac{1-i\cot\varphi}{2\alpha-i\cot\varphi}} e^{-\frac{\csc^2\varphi\eta^2}{4\alpha-2i\cot\varphi} + \frac{i\eta^2\cot\varphi}{2}} \right|^2 d\eta \\ &= \sqrt{\frac{1+\cot^2\varphi}{4\alpha^2+\cot^2\varphi}} \int_{-\infty}^{\infty} \eta^2 e^{-\frac{8\alpha\csc^2\varphi\eta^2}{16\alpha^2+4\cot^2\varphi}} d\eta \\ &= \frac{\sqrt{\pi}(4\alpha^2\sin^2\varphi+\cos^2\varphi)}{2(2\alpha)^{\frac{3}{2}}}. \end{aligned} \quad (48)$$

Especially, when $\varphi = \frac{\pi}{2}$ in Equation (48), we have

$$\int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^{\frac{\pi}{2}}\{e^{-\alpha t^2}\}(\eta) \right|^2 d\eta = \sqrt{\frac{\pi\alpha}{2}}, \quad (49)$$

and for $\varphi = \frac{\pi}{4}$,

$$\int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^{\frac{\pi}{4}}\{e^{-\alpha t^2}\}(\eta) \right|^2 d\eta = \frac{\sqrt{\pi}(4\alpha^2+1)}{4(2\alpha)^{\frac{3}{2}}} \quad (50)$$

From Equations (49) and (50), it is obvious that

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^{\frac{\pi}{2}}\{f\}(\eta) \right|^2 d\eta \int_{-\infty}^{\infty} v^2 \left| \mathcal{F}^{\frac{\pi}{4}}\{f\}(v) \right|^2 dv \\ &= \frac{\pi^{\frac{3}{2}}(4\alpha^2+1)}{16\alpha^{\frac{3}{2}}} \\ &\geq \frac{\pi}{16\alpha} = \frac{|\sin\frac{\pi}{4}|^2}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2 \end{aligned} \quad (51)$$

This shows that uncertainty principle (17) holds right. Based on Equation (51), equality of the principle (17) is achieved at Gaussian function, which the parameter α satisfies $\frac{\pi^{\frac{3}{2}}(4\alpha^2+1)}{16\alpha^{\frac{3}{2}}} = \frac{\pi}{16\alpha}$

Example 2. Suppose the Gabor filter is defined as:

$$f(t) = e^{iu_0t} e^{-\alpha t^2}, \quad \alpha > 0. \quad (52)$$

Making use of (4) gives

$$\begin{aligned} & \mathcal{F}^{\varphi}\{e^{iu_0t} e^{-\alpha t^2}\}(\eta) \\ &= B_{\varphi} \int_{-\infty}^{\infty} e^{iu_0t} e^{-\alpha t^2} e^{i(t^2+\eta^2)\frac{\cot\varphi}{2}-it\eta\csc\varphi} dt \\ &= B_{\varphi} \int_{-\infty}^{\infty} e^{-(\alpha-i\frac{\cot\varphi}{2})t^2+i(u_0-\eta\csc\varphi)t} dt e^{i\eta^2\frac{\cot\varphi}{2}} \end{aligned} \quad (53)$$

Applying Equations (42) to (53) results in

$$\mathcal{F}^\varphi \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) = \sqrt{\frac{1-i \cot \varphi}{2\pi}} \sqrt{\frac{\pi}{\alpha-i \frac{\cot \varphi}{2}}} \times e^{-\frac{(u_0-\eta \csc \varphi)^2}{4(\alpha-i \frac{\cot \varphi}{2})} + i\eta^2 \frac{\cot \varphi}{2}} \quad (54)$$

Simplifying (54) yields

$$\mathcal{F}^\varphi \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) = \sqrt{\frac{1-i \cot \varphi}{2\alpha-i \cot \varphi}} \cdot e^{-\frac{(u_0-\eta \csc \varphi)^2}{4\alpha-2i \cot \varphi} + i\eta^2 \frac{\cot \varphi}{2}} \quad (55)$$

Remark 3.

1. If $\varphi = \frac{\pi}{2}$, Equation (55) changes to

$$\mathcal{F}^{\frac{\pi}{2}} \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) = \sqrt{\frac{1}{2\alpha}} e^{-\frac{(u_0+\eta)^2}{4\alpha}}$$

2. If $\varphi = \frac{\pi}{4}$ in (55), we have

$$\mathcal{F}^{\frac{\pi}{4}} \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) = \sqrt{\frac{1-i}{2\alpha-i}} e^{-\frac{(u_0-\frac{\eta}{\sqrt{2}})^2}{4\alpha-2i} - \frac{i\eta^2}{2}}$$

Further, simple computation shows that

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^\varphi \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) \right|^2 d\eta \\ &= \int_{-\infty}^{\infty} \eta^2 \left| \sqrt{\frac{1-i \cot \varphi}{2\alpha-i \cot \varphi}} e^{-\frac{(u_0-\eta \csc \varphi)^2}{4\alpha-2i \cot \varphi} + i\eta^2 \frac{\cot \varphi}{2}} \right|^2 d\eta \\ &= \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \int_{-\infty}^{\infty} \eta^2 \left| e^{-\frac{(u_0-\eta \csc \varphi)^2}{4\alpha-2i \cot \varphi}} \right|^2 d\eta \\ &= \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \int_{-\infty}^{\infty} \eta^2 e^{-\frac{8\alpha(u_0-\eta \csc \varphi)^2}{16\alpha^2+4\cot^2 \varphi}} d\eta \quad (56) \end{aligned}$$

By the change of variable $v = u_0 - \eta \csc \varphi$, we easily get

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^\varphi \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) \right|^2 d\eta \\ &= \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \int_{-\infty}^{\infty} \frac{(v-u_0)^2}{\csc^2 \varphi} e^{-\frac{2\alpha v^2}{4\alpha^2+\cot^2 \varphi}} \frac{dv}{\csc \varphi} \\ &= \frac{1}{\csc^3 \varphi} \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \int_{-\infty}^{\infty} (v^2 - 2u_0 v + u_0^2) e^{-\frac{2\alpha v^2}{4\alpha^2+\cot^2 \varphi}} dv \end{aligned}$$

Applying Equation (47) to above expression we get

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^\varphi \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) \right|^2 d\eta \\ &= \frac{1}{\csc^3 \varphi} \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \int_{-\infty}^{\infty} v^2 e^{-\frac{2\alpha v^2}{4\alpha^2+\cot^2 \varphi}} dv \\ &\quad - \frac{2}{\csc^3 \varphi} \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \int_{-\infty}^{\infty} u_0 v e^{-\frac{2\alpha v^2}{4\alpha^2+\cot^2 \varphi}} dv \\ &\quad + \frac{1}{\csc^3 \varphi} \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \int_{-\infty}^{\infty} u_0^2 e^{-\frac{2\alpha v^2}{4\alpha^2+\cot^2 \varphi}} dv \\ &= \frac{1}{\csc^3 \varphi} \sqrt{\frac{1+\cot^2 \varphi}{4\alpha^2+\cot^2 \varphi}} \left(\frac{\sqrt{\pi}}{2 \left(\frac{2\alpha}{4\alpha^2+\cot^2 \varphi} \right)^{\frac{3}{2}}} \right. \\ &\quad \left. - u_0^2 \sqrt{\frac{2\alpha\pi}{4\alpha^2+\cot^2 \varphi}} \right) \\ &= \frac{\sqrt{\pi}(4\alpha^2+\cot^2 \varphi)}{2 \csc^2 \varphi (2\alpha)^{\frac{3}{2}}} \\ &\quad - u_0^2 \frac{\sqrt{2\alpha\pi}}{\csc^2 \varphi (4\alpha^2+\cot^2 \varphi)}, \quad (57) \end{aligned}$$

where we have used integral of an odd function equal to zero.

Substituting $\varphi = \frac{\pi}{2}$ in (57) yields

$$\int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^{\frac{\pi}{2}} \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) \right|^2 d\eta = \sqrt{\frac{\alpha\pi}{2}} - u_0^2 \sqrt{\frac{\pi}{(2\alpha)^3}} \quad (58)$$

For $\varphi = \frac{\pi}{4}$, we have

$$\int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^{\frac{\pi}{4}} \{e^{iu_0 t} e^{-\alpha t^2}\}(\eta) \right|^2 d\eta = \frac{(8\alpha^2+1)\sqrt{\pi}}{8(2\alpha)^{\frac{3}{2}}} - \frac{u_0^2}{4\alpha^2+1} \sqrt{\frac{\alpha\pi}{8}} \quad (59)$$

From Equations (58) and (59), it is obvious that

$$\begin{aligned} & \int_{-\infty}^{\infty} \eta^2 \left| \mathcal{F}^{\frac{\pi}{2}} \{f\}(\eta) \right|^2 d\eta \int_{-\infty}^{\infty} v^2 \left| \mathcal{F}^{\frac{\pi}{4}} \{f\}(v) \right|^2 dv \\ &= \left(\sqrt{\frac{\alpha\pi}{2}} - u_0^2 \sqrt{\frac{\pi}{(2\alpha)^3}} \right) \left(\frac{(8\alpha^2+1)\sqrt{\pi}}{8(2\alpha)^{\frac{3}{2}}} - \frac{u_0^2}{4\alpha^2+1} \sqrt{\frac{\alpha\pi}{8}} \right) \\ &\geq \frac{\pi}{16\alpha} \\ &= \frac{|\sin \frac{\pi}{2}|^2}{4} \left(\int_{-\infty}^{\infty} |f(t)|^2 dt \right)^2, \quad \alpha > \frac{1}{\sqrt{8}} \quad (60) \end{aligned}$$

4. CONCLUSIONS

In this study, we have recalled a definition of the FrFT and collected its essential properties. We provided the alternative proofs of the uncertainty principles related to this transformation, whose proofs are simpler than those found in the existing

literature. We also established the general form of the uncertainty principle related to the proposed FrFT. The uncertainty principles formulated in this paper play a crucial role in the development of harmonic analysis as well as in understanding the behaviour of signals related to integral transforms. In the future work, we shall extend these uncertainty principles in the context of the coupled fractional Fourier transform (Shah et al., 2021; Kamalakkannan et al., 2021) and multidimensional fractional Fourier transform (Kamalakkannan and Roopkumar, 2019).

5. ACKNOWLEDGEMENT

The present work is partially supported by Thematic Research Group (TRG) 2026 (No. 1167/UN4.1.7/PT.01.03/2026) of Hasanuddin University, Indonesia.

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