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Research Paper

# On Distance Vertex Irregular Total $k$-Labeling

Dian Eka Wijayanti[1,4]*, Noor Hidayat[1], Diari Indriati[2], Abdul Rouf Alghofari[1], Slamin[3]

[1]Department of Mathematics, Universitas Brawijaya, Malang, 65145, Indonesia
[2]Department of Mathematics, Universitas Sebelas Maret, Surakarta, 57126, Indonesia
[3]Department of Informatics, University of Jember, Jember, 68121, Indonesia
[4]Department of Mathematics, Universitas Ahmad Dahlan, Kampus IV UAD, Yogyakarta, 55166, Indonesia
*Corresponding author: dian@math.uad.ac.id

**Abstract**
Let $H$= ($T$,$S$), be a finite simple graph, $T(H)$= $T$ and $S(H)$= $S$, respectively, are the sets of vertices and edges on $H$. Let $\sigma$:$T\cup S\rightarrow 1,2,\cdots,k$, be a total $k$-labeling on $H$ and $w_\sigma$ ($x$), be a weight of $x\in T$ while using $\sigma$ labeling, which is evaluated based on the total number of all vertices labels in the neighborhood $x$ and its incident edges. If every $x\in T$ has a different weight, then $\sigma$ is a distance vertex irregular total $k$-labeling (DVITL). Total distance vertex irregularity strength of $H$ ($tdis(H)$ is defined as the least $k$ for which $H$ has a DVITL. Our research investigates the DVITL of the path ($P_r$) and cycle ($C_r$) graphs. We establish a lower bound and then calculate the precise value of $tdis(P_r)$ and $tdis(C_r)$.

**Keywords**
Irregular Labeling, Distance Labeling, Total Labeling, Irregularity Strength

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## 1. INTRODUCTION

Consider a simple finite graph $H$= ($T$,$S$), $T(H)$= $T$ and $S(H)$= $S$, respectively, are the sets of vertices and edges on $H$. Wallis (2001) defines the labeling of the graph as a mapping from the set of graph elements to the set of non negative integers. The three sets that are often used as domains are $T$ (Vertex Labelings (VL)), $S$ (Edge Labelings (EL)), and $T\cup S$ (Total Labelings (TL)). The recent analysis of graph labelings can be seen in Gallian (2018).

One type of VL was Distance Vertex Magic Labeling (DV ML), introduced by Miller et al. (2003) as 1-vertex-magic labeling. In this labeling, the vertex weight is evaluated using the total number of labels for every vertex in the neighborhood of distance-1 of $x$. Chartrand et al. (1988) were the first to discuss Irregular Labeling (IL) in 1988. In IL, the vertex weight is evaluated using the total number of its incidence edges labels. Finding the least of maximum labels to assign to the graph edges while maintaining distinct weights for each vertex is the goal of IL. The term irregularity strength of $H$, symbolize by $s(H)$, stand for the least of maximum label among all of the possible irregular assignments of $H$. Bača et al. (2007) established a novel type of IL known as Irregular Total Labeling (ITL), which was motivated by the concepts of irregularity strength and TL. For graph $H(T,S)$, a labeling $\sigma$: $T\cup S\rightarrow 1,2,\cdots,k$ is a Vertex Irregular Total Labeling (VITL) if for every $x,y\in T, x\neq y$ then $w_\sigma(x)\neq w_\sigma(y)$. For every $x\in T$, $w_\sigma(x)$ is defined as

$$w_\sigma(x) = \sigma(x) + \sum_{ux\in S} \sigma(ux)$$

The labeling $\sigma$ is an Edge Irregular Total Labeling (EITL) if for every $xy$, $uv\in S$, $xy\neq uv$ then $w_\sigma(xy)\neq w_\sigma(uv)$. For every $xy\in E$, $w_\sigma(xy)$ is defined as

$$w_\sigma(xy) = \sigma(xy) + \sigma(x) + \sigma(y)$$

Furthermore, motivated by the definition of DVML and ITL, Slamin (2017) introduces new labeling by defining a Distance Vertex Irregular Labeling (DVIL) of graph $H(T,S)$, as a function $\sigma$:$T\cup S\rightarrow 1,2,\cdots,$k which makes each $x\in T$, has a unique weight. The distance vertex irregularity strength of $H$ denote by $dis(H)$ and the weight of $x\in T$ while using $\sigma$ labeling defined as

$$w_\sigma(x) = \sum_{u\in N(x)} \sigma(u)$$

Bong et al. (2017) continued the research by complete the concept of $dis(C_r)$ and $dis(W_r)$, while Bača et al. (2018) investigate inclusive DVIL.

Motivated by the definition of ITL by Baskoro et al. (2010), Bača et al. (2007), Indriati et al. (2013) and DVIL by Slamin (2017), Bača et al. (2018), Sugeng et al. (2021), Wijayanti et al. (2021), Wijayanti et al. (2023) introduce a new graph labeling based on both ITL and DVIL, named Distance Vertex Irregular Total $k$-Labeling (DVITL) and denote the total distance vertex irregularity strength of $H(T,S)$ by $tdis(H)$. The domain of this labeling is $T \cup S$ and the weights of vertices are count based on distance.

Considering that the idea of distance labeling is identical, to obtain the formula of lower bound of $tdis(H)$, Wijayanti et al. (2023) modified the formula of lower bound of $dis(H)$ (Slamin, 2017) and aply it to DVITL. Moreover, Wijayanti et al. (2021) and Wijayanti et al. (2023) investigate the *tdis* of fans, wheels, and some corona product graphs. In this paper, we apply the DVITL to path ($P_r$) and cycle ($C_r$) graphs, and investigate the lower bound and the precise value of $tdis(P_r)$ and $tdis(C_r)$.

## 2. PRELIMINARIES

Various authors with varied terminology introduced and researched the concept of distance labeling. Miller et al. (2003) referred to the labeling as 1-vertex magic, while Sugeng et al. (2009) referred to it as distance magic labeling. Beena (2009) first described the idea as $\sigma$ and $\sigma$' labeling. O'Neal and Slater (2011) were the first to introduce the theory of $D$-vertex magic labeling, which also encompasses the concept of 1-vertex magic labeling, distance magic labeling, $\sigma$ and $\sigma$'-labeling as particular cases. The survey of the distance magic graphs can be seen in Arumugam et al. (2011).

Chartrand et al. (1988) first introduced Irregular labeling in 1988, which developed very rapidly since then. Various types of irregular labeling such as VITL (Anholcer et al., 2009; Bača et al., 2007; Indriati et al., 2016; Baskoro et al., 2010), EITL (Anholcer and Palmer, 2012;) Indriati et al., 2013), totally ITL (Indriati et al., 2020; Marzuki et al., 2013), DVIL (Bong et al., 2017; Novindasari et al., 2016; Slamin, 2017; Sugeng et al., 2021; Susanto et al., 2022a; Susanto et al., 2022b), and DVITL (Wijayanti et al., 2021; Wijayanti et al., 2023). Wijayanti et al. (2021) define the basic concept of DVITL and $tdis(G)$ and also research some necessary and sufficient conditions for the existence of DVITL (Wijayanti et al., 2023).

**Definition 1** *Let H(T,S) be a finite simple graph with* $|T|= r$ *and* $|S|= s$. *A DVITL of H is a function* $\sigma : T \cup S \rightarrow 1,2,\cdots,k$ *such that every* $x \in T$ *has a unique weight. While using labeling* $\sigma$, *the weight of* $x \in T$ *is defined as*

$$w_\sigma(x) = \sum_{u \in N(x)} \sigma(u) + \sum_{ux \in I(x)} \sigma(ux) \quad (1)$$

The total distance vertex irregularity strength of $H$, denote by $tdis(H)$ and defined as the least of maximum label among all of the possible DVITL of $H$.

The examples of two DVITL on $C_4$ and $tdis(C_4)$ are as seen in Figure 1, where the integers inside blue boxes are $w_\sigma(x_i)$; $i= 1,\cdots,4$, and the integers outside blue boxes are $\sigma(x_i)$; $i= 1,\cdots,4$ and $\sigma(x_i x_{i+1})$; $i= 1,2,3$, $\sigma(x_1 x_4)$.

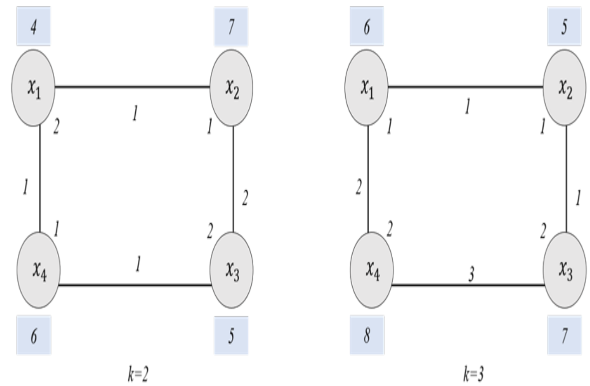


**Figure 1.** Examples of Two DVITL on $C_4$, where $tdis(C_4)= 2$

The following lemma will gives the lower bound of $tdis(G)$.

**Lemma 1** *Let H(T,S) be a finite simple graph without isolated vertex, with* $\phi \leq deg(x) \leq \Phi$, *for all* $x \in T$. *Then,*

$$tdis(H) \geq \left\lceil \frac{|T| - 1 + 2\phi}{2\Phi} \right\rceil \quad (2)$$

*Proof.* Let $H(T,S)$ be a finite simple graph without isolated vertex. Since $\phi \leq \deg(x) \leq \Phi$ for all $x \in T$, the minimal weight of all vertices in $V(G)$ is $2\phi$, that is when we assign the label 1 to all of the neighbors and incidence edges of a vertex of degree $\phi$. The weight of all vertices must be different from one another, therefore, the minimal value of the largest weight is obtained when the weight sequence forms an arithmetic progression. Given that $H$ contains $|T|$ vertices, we obtain the minimum of the maximum weight is $2\phi+|T|-1$. This weight must belong to the vertex of degree $\Phi$, and is the sum of the $2\Phi$ integers. Hence, the largest label that contributes to this weight needs to be greatest than or equal to $\left\lceil \frac{|T|-1+2\phi}{2\Phi} \right\rceil$.

In the following subsection, we define the DVITL and determine the *tdis* of path and cycle graphs then formulate it through theorems. To prove the theorems, we use Lemma 1 to obtain the lower bound of the *tdis* and show that the DVITL exists for path and cycle graphs, where $k$ is equal to the lower bound.

### 2.1 The DVITL and The *tdis* of The Path Graph

The following theorems present the *tdis* of the path graph.

**Theorem 1** *Let r be a non-negative integer and* $P_r$ *is a path graph with r vertices, then*

$$tdis(P_r) = \left\{ \begin{array}{ll} \left\lceil \frac{r+1}{4} \right\rceil & , r \neq 2, 3; \\ 2 & , r = 2, 3; \end{array} \right| \quad (3)$$

*Proof.* Let $P_r$ $(T,S)$ is a path graph, $T(P_r)$ be a vertex set and $S(P_r)$ be an edge set of $P_r$, where $T(P_r)= \{x_1,x_2,\cdots,x_r\}$, $S(P_r)= \{x_1x_2,x_2x_3,\cdots,x_{r-1}x_r\}$,$|T(P_r)|= r$, $|S(P_r)|= r$-1. The maximum degree of $P_r$, $\Phi$= 2 and the minimum degree of $P_r$, $\phi$= 1. We prove this theorem by using Lemma 1 to obtain the lower bound of $tdis(P_r)$ and prove the equality of $tdis(P_r)$ in Equation (3). By subtituting the value of $r$, $\Phi$ and $\phi$ to Inequation (2), we obtain

$$tdis(P_r) \geq \left\lceil \frac{r+1}{4} \right\rceil \quad (4)$$

Since Inequation (4) has given the lower bound of $tdis(P_r)$, showing that $P_r$ has a DVITL with $k= \lceil \frac{r+1}{4} \rceil$ for r≠2,3 and $k$= 2 for $r$= 2,3 is sufficient to prove Equality (3). We do the proving by considering four cases.

**Case 1** : *For r≤8.*

For $r$≤8, We assign the following labels to the vertices and edges of $P_r$ as follows,

From Table 1, It is obvious that each vertex of $P_r$ has a unique weight, where $k= \lceil \frac{r+1}{4} \rceil$ for $r$≠2,3 and $k$= 2 for $r$= 2,3.

**Case 2** : *For r ≡0,1 (mod 4),r>8.*

Define $\sigma$, the DVITL of $P_r$ as follows

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j-1}{2} \right\rceil & , j = 2, \cdots, \left\lceil \frac{r-1}{2} \right\rceil + 1; \\ \left\lceil \frac{r-j+2}{2} \right\rceil & , j = \left\lceil \frac{r-1}{2} \right\rceil + 2, \cdots, r; \\ 1 & , j = 1; \end{cases} \quad (5)$$

$$\sigma(x_jx_{j+1}) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil & , j = 1, \cdots, \left\lceil \frac{r-1}{2} \right\rceil + 1; \\ \left\lceil \frac{r-j}{2} \right\rceil & , j = \left\lceil \frac{r-j}{2} \right\rceil + 2, \cdots, r-1; \end{cases} \quad (6)$$

All vertices and edges labels are less than or equal to $\lceil \frac{r+1}{4} \rceil$. Based on Equation (1), we have

$$w_\sigma(x_j) = \sum_{x_k \in N(x_j)} \sigma(x_k) + \sigma(x_jx_k)$$

Observe that,

$w_\sigma(x_1)= \sigma(x_2)+\sigma(x_1x_2)$= 1+1= 2,

$w_\sigma(x_r)= \sigma(x_{r-1})+\sigma(x_{r-1}x_r)= \left\lceil \frac{r-(r-1)+2}{2} \right\rceil + \left\lceil \frac{r-(r-1)}{2} \right\rceil$= 2+1= 3,

$w_\sigma(x_2)= \sigma(x_1)+\sigma(x_1x_2)+\sigma(x_2x_3)+\sigma(x_3)$= 1+1+1+1= 4,

$w_\sigma(x_r)= \sigma(x_{r-2})+\sigma(x_{r-2}x_{r-1})+\sigma(x_{r-1}x_r)+\sigma(x_r)$= 2+1+1+1= 5,

$w_\sigma(x_3)= \sigma(x_2)+\sigma(x_2x_3)+\sigma(x_3x_4)+\sigma(x_4)$= 1+1+2+2= 6,

$w_\sigma(x_4)= \sigma(x_3)+\sigma(x_3x_4)+\sigma(x_4x_5)+\sigma(x_5)$= 1+2+2+2= 7,

$w_\sigma(x_{r-2})= \sigma(x_{r-3})+\sigma(x_{r-3}x_{r-2})+\sigma(x_{r-2}x_{r-1})+\sigma(x_{r-1})$= 3+2+1 +2= 8,

$\vdots$

$$w_\sigma\left(x_{\frac{r}{2}+1}\right) = \sigma\left(x_{\frac{r}{2}}\right) + \sigma\left(x_{\frac{r}{2}}x_{\frac{r}{2}+1}\right) + \sigma\left(x_{\frac{r}{2}+1}x_{\frac{r}{2}+2}\right) + \sigma\left(x_{\frac{r}{2}+2}\right)$$
$$= \left\lceil \frac{r}{4} \right\rceil + \left\lceil \frac{r+2}{4} \right\rceil + \left\lceil \frac{r+4}{4} \right\rceil + \left\lceil \frac{r+4}{4} \right\rceil 4\frac{r}{4} + \left\lceil \frac{2}{4} \right\rceil + \left\lceil \frac{4}{4} \right\rceil + \left\lceil \frac{4}{4} \right\rceil = r+3.$$

Hence, the vertices weights set is $\{w_\sigma(x_1), \cdots, w_\sigma(x_r)\}$= {4,5, $\cdots$,$r$+3} for $i$∈{1,2,$\cdots$,$r$}. Each vertex weight is unique, and $k = \lceil \frac{r+3}{4} \rceil$.

**Case 3** : *For r ≡2 (mod 4),r>8.*

Define $\sigma$, the DVITL of $P_r$ as follows

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j-1}{2} \right\rceil & , j = 2, \cdots, \frac{r}{2} + 2; \\ \left\lceil \frac{r-j}{2} \right\rceil & , j = \frac{r}{2} + 3, \cdots, r-2; \\ 1 & , j = 1; \\ 2 & , j = r-1, r; \end{cases} \quad (7)$$

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil & , j = 1, \cdots, \frac{r}{2} + 1; \\ \left\lceil \frac{r-3}{4} \right\rceil & , j = \frac{r}{2} + 2; \\ \left\lceil \frac{r-j+2}{2} \right\rceil & , j = \frac{r}{2} + 3, \cdots, n-3; \\ 1 & , j = r-2, r-1; \end{cases} \quad (8)$$

Obviously, every vertex and edge of $P_r$ has a label that is not greater than $\frac{r-2}{4} + 1 = \lceil \frac{r+1}{4} \rceil$. Similarly with Case 2, the collection of all vertices weights is $\{w_\sigma(x_1), \cdots, w_\sigma(x_r)\} = \{2, 3, \cdots, r+1\}$ for $j \in \{1, 2, \cdots, r\}$. The weights of all vertices are distinct and $k = \lceil \frac{r+1}{4} \rceil$.

**Table 1.** The DVITL for $P_r$,$r\leq 8$

| Graph | $\sigma(x_j)$ | | | | | | | | $\sigma(x_j x_k)$ | | | | | | | $W_\sigma(x_j)$ | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ | $x_1x_2$ | $x_2x_3$ | $x_3x_4$ | $x_4x_5$ | $x_5x_6$ | $x_6x_7$ | $x_7x_8$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_7$ | $x_8$ |
| $P_1$ | 1 | | | | | | | | | | | | | | | 1 | | | | | | | |
| $P_2$ | 2 | 1 | | | | | | | 1 | | | | | | | 2 | 3 | | | | | | |
| $P_3$ | 1 | 1 | 1 | | | | | | 1 | 2 | | | | | | 2 | 5 | 3 | | | | | |
| $P_4$ | 1 | 1 | 2 | 1 | | | | | 1 | 1 | 1 | | | | | 2 | 5 | 4 | 3 | | | | |
| $P_5$ | 1 | 1 | 1 | 2 | 1 | | | | 1 | 1 | 2 | 1 | | | | 2 | 4 | 6 | 5 | 3 | | | |
| $P_6$ | 2 | 1 | 2 | 1 | 2 | 1 | | | 1 | 2 | 1 | 1 | 1 | | | 2 | 7 | 5 | 6 | 4 | 3 | | |
| $P_7$ | 1 | 1 | 1 | 2 | 2 | 2 | 2 | | 1 | 1 | 1 | 2 | 2 | 1 | | 2 | 4 | 5 | 6 | 8 | 7 | 3 | |
| $P_8$ | 1 | 1 | 1 | 2 | 3 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 2 | 2 | 1 | 2 | 4 | 6 | 8 | 7 | 9 | 5 | 3 |

**Case 4** *: For r ≡3 (mod 4) and r>8.*

Define $\sigma$, the DVITL of $P_r$ as follows

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j-1}{2} \right\rceil & , j = 2, \cdots, \frac{r+1}{2}+1; \\ \left\lceil \frac{r+1}{4} \right\rceil & , j = \frac{r+1}{2}+2; \\ \left\lceil \frac{r-j}{2} \right\rceil & , j = \frac{r+1}{2}+3, \cdots, r-2; \\ 1 & , j = 1; \\ 2 & , j = r-1, r; \end{cases} \tag{9}$$

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil & , j = 1, \cdots, \frac{r+1}{2}; \\ \left\lceil \frac{r-j+1}{2} \right\rceil & , j = \frac{r+1}{2}+1, \frac{r+1}{2}+2; \\ \left\lceil \frac{r-j+2}{2} \right\rceil & , j = \frac{r+1}{2}+3, \cdots, r-3; \\ 1 & , j = r-2, r-1; \end{cases} \tag{10}$$

It is clear from Equation (10) that none of the labels for the vertices and edges are higher than $\left\lceil \frac{r+1}{4} \right\rceil$. Similarly with Case 2 and 3, for $i\in\{1, 2, \cdots, r\}$, we obtain $\{w_\sigma(x_1), \cdots, w_\sigma(x_r)\} = \{2, 3, \cdots, r+1\}$. Every $x_j \in V(P_r)$ has a different weight when $k = \left\lceil \frac{r+1}{4} \right\rceil$.

## 2.2 The DVITL and the *tdis* of the Cycle Graph

Cycle graph admits the DVITL and $tdis(C_r)$ is determined below.

**Theorem 2** *Let $r\geq 3$ be an integer, and $C_r$ be a cycle graph with r vertices, then*

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{r+3}{4} \right\rceil & , r \neq 5; \\ \left\lceil \frac{r+3}{4} \right\rceil + 1 & , r = 5 \end{cases} \tag{11}$$

*Proof.* Let $C_r(T,S)$ be a cycle graph with $r$ vertices, $T(C_r)$ be a vertex set, $S(C_r)$ be an edge set of $C_r$ where $T(C_r)$= $\{x_1, x_2, \cdots, x_r\}$,$S(C_r)$= $\{x_1x_2, x_2x_3, \cdots, x_{r-1}x_r, x_rx_1\}$, $|T(C_r)|$ =$|S(C_r)|$= $r$. Since $C_r$ is 2-regular graph, the minimum is equal to the maximum, which is $\phi$= $\Phi$= 2. The lower bound of $tdis(C_r)$, can be obtained from Lemma 1. Similarly with path graph, by subtituting the value of $r$,$\Phi$ and $\phi$ to Inequation (2), we obtain,

$$tdis(C_r) \geq \left\lceil \frac{r+3}{4} \right\rceil \tag{12}$$

Since Inequation (12) has given the lower bound of $tdis(C_r)$, showing that $C_r$ has a DVITL with $k$= $\left\lceil \frac{r+3}{4} \right\rceil$ is sufficient to prove Equality (11). We do the proving by considering five cases.

**Case 1** : *For* $r\in\{3, 4, 5, 6\}$,

Label all vertices and edges of $C_r$ as in Table 2,

Thus, each vertex weight of $C_r$ is different from the other. For $r\in\{3,4,6\}$, $tdis(C_r)$=$\left\lceil \frac{r+3}{4} \right\rceil$ and for $r$= 5,$tdis(C_r)$= $\left\lceil \frac{r+3}{4} \right\rceil$+1.

**Case 2** : *For* $r \equiv 0 \ (mod\ 4)$, $r \geq 7$

Define $\sigma$, a DVITL of $C_r$ as follows

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil & , j = 1, \cdots, \frac{r}{2}+2; \\ \left\lceil \frac{r-j+1}{2} \right\rceil & , j = \frac{r}{2}+3, \cdots, r; \end{cases} \tag{13}$$

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j+1}{2} \right\rceil & , j = 1, \cdots, \frac{r}{2}+1, k = j+1; \\ \left\lceil \frac{r}{4} \right\rceil & , j = \frac{r}{2}+2, k = j+1; \\ \left\lceil \frac{r-j+3}{2} \right\rceil & , j = \frac{r}{2}+3, \cdots, r-1, k = j+1; \\ 1 & , j = r, k = 1; \end{cases} \tag{14}$$

As can be observed, each vertex and edge label is not greater than $\left\lceil \frac{r+3}{4} \right\rceil$ ($\sigma(x_j)$ for $j$= $\frac{r}{2}+1, \frac{r}{2}+2$ and $\sigma(x_jx_{(j+1)})$ for $j$= $(\frac{r}{2}, \frac{r}{2}+1, \frac{r}{2}+2)$. From Equation (1), we have

**Table 2.** The DVITL for $C_r, r \in \{3,4,5,6\}$

| Graph | $\sigma(x_j)$ | | | | | | $\sigma(x_jx_k)$ | | | | | | | | | $w_\sigma(x_j)$ | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ | $x_1x_2$ | $x_2x_3$ | $x_3x_1$ | $x_3x_4$ | $x_4x_1$ | $x_4x_5$ | $x_5x_1$ | $x_5x_6$ | $x_6x_1$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| $P_1$ | 1 | 1 | 2 | | | | 2 | 2 | 1 | | | | | | | 6 | 7 | 5 | | | |
| $P_2$ | 2 | 1 | 2 | 1 | | | 1 | 1 | | 2 | 1 | | | | | 4 | 6 | 5 | 7 | | |
| $P_3$ | 1 | 1 | 3 | 2 | 1 | | 1 | 2 | | 3 | | 2 | 1 | | | 4 | 7 | 8 | 9 | 6 | |
| $P_4$ | 1 | 1 | 2 | 2 | 2 | 1 | 1 | 2 | | 3 | | 3 | | 1 | 1 | 4 | 6 | 8 | 10 | 7 | 5 |

$$w_\sigma(x_j) = \sum_{x_k \in N(x_j)} \sigma(x_k) + \sigma(x_jx_k)$$

Observe that,

$w_\sigma(x_1) = \sigma(x_r) + \sigma(x_rx_1) + \sigma(x_2) + \sigma(x_1x_2) = 1+1+1+1 = 4$,

$w_\sigma(x_r) = \sigma(x_1) + \sigma(x_rx_1) + \sigma(x_{r-1}) + \sigma(x_{r-1}x_r) = 1+1+2+1 = 5$,

$w_\sigma(x_2) = \sigma(x_1) + \sigma(x_1x_2) + \sigma(x_2x_3) + \sigma(x_3) = 1+1+2+2 = 6$,

$w_\sigma(x_3) = \sigma(x_2) + \sigma(x_2x_3) + \sigma(x_3x_4) + \sigma(x_4) = 1+2+2+2 = 7$,

$w_\sigma(x_{r-1}) = \sigma(x_{r-2}) + \sigma(x_{r-2}x_{r-1}) + \sigma(x_{r-1}x_r) + \sigma(x_r) = 2+3+2+1 = 8$,

$w_\sigma(x_{r-2}) = \sigma(x_{r-3}) + \sigma(x_{r-3}x_{r-2}) + \sigma(x_{r-2}x_{r-1}) + \sigma(x_{r-1}) = 2+3+3+1 = 9$,

$\vdots$

$$w_\sigma\left(x_{\frac{r}{2}+1}\right) = \sigma\left(x_{\frac{r}{2}}\right) + \sigma\left(x_{\frac{r}{2}}x_{\frac{r}{2}+1}\right) + \sigma\left(x_{\frac{r}{2}+1}x_{\frac{r}{2}+2}\right) + \sigma\left(x_{\frac{r}{2}+2}\right)$$
$$= \left\lceil \tfrac{r}{4} \right\rceil + \left\lceil \tfrac{r+2}{4} \right\rceil + \left\lceil \tfrac{r+4}{4} \right\rceil + \left\lceil \tfrac{r+4}{4} \right\rceil 4\tfrac{r}{4} + \left\lceil \tfrac{2}{4} \right\rceil + \left\lceil \tfrac{4}{4} \right\rceil + \left\lceil \tfrac{4}{4} \right\rceil = r+3.$$

Hence, the vertices weights set is $\{w_\sigma(x_1), \cdots, w_\sigma(x_r)\} = \{4, 5, \cdots, r+3\}$ for $i \in \{1, 2, \cdots, r\}$. Each vertex weight is unique, and $k = \left\lceil \frac{r+3}{4} \right\rceil$.

**Case 3** : *For* $r \equiv 1 \ (mod\ 4), r \geq 7$

Define $\sigma$, a DVITL of $C_r$ as follows

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil & , j = 1, \cdots, \frac{r+1}{2}+1; \\ \left\lceil \frac{r+3}{4} \right\rceil & , j = \frac{r+1}{2}+2; \\ \left\lceil \frac{r-j+1}{2} \right\rceil & , j = \frac{r+1}{2}+3, \cdots, r. \end{cases} \quad (15)$$

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j+1}{2} \right\rceil & , j = 1, \cdots, \frac{r+1}{2}, k = j+1; \\ \left\lceil \frac{r-1}{4} \right\rceil & , j = \frac{r+1}{2}+2, k = j+1; \\ \left\lceil \frac{r-j+3}{2} \right\rceil & , j = \frac{r+1}{2}+1, \cdots, r-1, \\ & \quad j \neq \frac{r+1}{2}+2, k = j+1; \\ 1 & , j = r, k = 1; \end{cases} \quad (16)$$

The largest label of all vertices and edges is $\left\lceil \frac{r+3}{4} \right\rceil$. Similarly with Case 2, for $j \in \{1, 2, \cdots, r\}$, the set of every vertex weight is $\{w_\sigma(x_1), \cdots, w_\sigma(x_r)\} = \{4, 5, \cdots, r+3\}$. The weights of each vertex vary from one another, where $k = \left\lceil \frac{r+3}{4} \right\rceil$.

**Case 4** : *For* $r \equiv 2 \ (mod\ 4), r \geq 7$

Define $\sigma$, a DVITL of $C_r$ as follows,

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j}{2} \right\rceil & , j = 1, \cdots, \frac{r}{2}-1; \\ \left\lceil \frac{r+3}{4} \right\rceil & , j = \frac{r}{2}; \\ \left\lceil \frac{r-j+1}{2} \right\rceil & , j = \frac{r}{2}+1, \cdots, r; \end{cases} \quad (17)$$

$$\sigma(x_j) = \begin{cases} \left\lceil \frac{j+1}{2} \right\rceil & , j = 1, \cdots, \frac{r}{2}-2, k = j+1; \\ \left\lceil \frac{r-2}{4} \right\rceil & , j = \frac{r}{2}-1, k = j+1; \\ \left\lceil \frac{r-j+3}{2} \right\rceil & , j = \frac{r}{2}, \cdots, r-1, k = j+1 \\ 1 & , j = r, k = 1; \end{cases} \quad (18)$$

As can be observed, each vertex and edge label is not greater than $\frac{r-3}{4} + 1 = \left\lceil \frac{r+3}{4} \right\rceil$. According to Equation (1), we have

$$w_\sigma(x_j) = \sum_{vk \in N(x_j)} \sigma(x_k) + \sigma(x_jx_k)$$

Obviously, for $j \in \{1,2,\cdots,r\}$, the set of every vertex weight is $\{w_\sigma(x_1), \cdots, w_\sigma(x_r)\} = \{2, 3, \cdots, r+3\}$. The weights of each vertex vary from one another, where $k = \lceil \frac{r+3}{4} \rceil$.

**Case 5** : *For* $r \equiv 3 \ (mod\ 4)$, $r \geq 7$

Define $\sigma$, a DVITL of $C_r$ as follows,

$$\sigma(x_j) = \begin{cases} \lceil \frac{j}{2} \rceil & , j = 1, \cdots, \frac{r+1}{2} - 2; \\ \lceil \frac{r-j+1}{2} \rceil & , j = \frac{r+1}{2} - 1, \cdots, r; \end{cases} \tag{19}$$

$$\sigma(x_j) = \begin{cases} \lceil \frac{j+1}{2} \rceil & , j = 1, \cdots, \frac{r+1}{2} - 3, k = j+1; \\ \lceil \frac{r-3}{4} \rceil & , j = \frac{r+1}{2} - 2, k = j+1; \\ \lceil \frac{r+3}{4} \rceil & , j = \frac{r+1}{2} - 1, k = j+1; \\ \lceil \frac{r-j+3}{2} \rceil & , j = \frac{r+1}{2}, \cdots, r-1, k = j+1 \\ 1 & , j = r, k = 1; \end{cases} \tag{20}$$

Each vertex and edge of $C_r$ has a label that is not greater than $\frac{r-3}{4} + 1 = \lceil \frac{r+1}{4} \rceil$ and by similar way with Case 2, 3, and 4, for $j \in \{1,2,\cdots,r\}$, all vertices weights set is $\{w_\sigma(x_1), \cdots, w_\sigma(x_r)\} = \{2, 3, \cdots, r+1\}$. The weights of each vertex vary from one another, where $k = \lceil \frac{r+3}{4} \rceil$.

## 3. CONCLUSIONS

In this research, we investigate DVITL, establish the lower bound, and identify the precise value of *tdis* for path ($P_r$) and cycle ($C_r$) graphs. For $P_r$, we obtain $tdis(P_r) = \lceil \frac{r+3}{4} \rceil$ for $r \neq 2,3$ and $tdis(P_r) = 2$ for $r = 2,3$. For ($C_r$), we obtain $tdis(C_r) = \lceil \frac{r+3}{4} \rceil$ for $r \neq 5$ and $tdis(C_r) = \lceil \frac{r+3}{4} \rceil + 1$ for $r = 5$. This paper concludes with the following open problems,

**Problems 1.** Determine $tdis(H)$, where $H$ is circulant graphs, generalized helm graphs and ladder graphs.

**Problems 2.** Investigate the DVITL and *tdis* of corona operations of arbitrary graphs.

**Problems 3.** Prove that all simple graphs have distance irregullar total $k$-labelings.

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