Generalized Hölder Inequality in Herz-Morrey Spaces with Variable Exponent

Hairur Rahman1, Corina Karim2

1Department of Mathematics, Islamic State University Maulana Malik Ibrahim Malang, Malang, 65144, Indonesia
2Department of Mathematics, Universitas Brawijaya, Malang, 65145, Indonesia
*Corresponding author: hairur@mat.uin-malang.ac.id

Abstract
The paper investigates the conditions for the generalized Hölder’s inequality with a variable exponent in Herz-Morrey spaces. The main results are based on the exponent functions p(·) and α(·). The proof of the first main result using the generalized Hölder’s inequality in Lebesgue spaces. The second main result of the paper is related to the weak space of the generalized Hölder’s inequality with a variable exponent in Herz-Morrey spaces. The theorems state the equivalence of certain conditions for the inequality. Mathematical proofs and analysis are providing to support the presented results for findings contribute to the understanding of Hölder’s inequalities in variable exponent spaces and their applications in Herz-Morrey spaces.

Keywords
Hölder Inequality, Herz-Morrey Spaces, Variable Exponent

1. INTRODUCTION

Variable exponent was initially examined in 1931 in the context of Lebesgue spaces by Orlicz (1931). Over the past thirty years, there has been a growing interest in the variable exponent, particularly in its applications in various fields such as electromechanical fluid dynamics, differential equations, among others. The concept of variable exponent led to the development of function spaces with variable exponents including Lebesgue spaces (Aoyama, 2009), Morrey spaces (Almeida et al., 2008; Fan, 2010; Mizuta, 2016; Mizuta and Ohno, 2015; Nogayama, 2010; Sultan et al., 2022; Wang and Liao, 2020; Wang and Wu, 2016). Nevertheless, the Hölder inequality also plays a significant role in this space. Some researchers have conducted research on the Herz-Morrey space (Chen et al., 2014; Dung et al., 2023; Izuki, 2010; Mizuta, 2016; Mizuta and Ohno, 2015; Nagayama, 2019; Sultan et al., 2022; Wang and Liao, 2020; Wang and Wu, 2016).
Theorem 1. Let $m \geq 2$. If $a(\cdot), a_i(\cdot) \in \mathbb{R}^n, 0 \leq \lambda, \lambda_i < \infty, p(\cdot), p_i(\cdot) \in PR^n$, and $0 < q_i, q_i < \infty$, for each $i = 1, \ldots, m$ then the subsequent statements are equivalent:

1. $\sum_{i=1}^{m} \frac{1}{a_i(\cdot)} \leq \frac{1}{a(\cdot)}, \sum_{i=1}^{m} \lambda = \lambda_i, \sum_{i=1}^{m} \frac{1}{p_i(\cdot)} \leq \frac{1}{p(\cdot)}$, and $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$.

2. $\left\| \prod_{i=1}^{m} f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q} \leq \left\| \prod_{i=1}^{m} f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q}$, for every $f_i \in MK^{a(\cdot),\lambda}_p(\cdot),q, i = 1, \ldots, m$.

Proof. (1 $\Rightarrow$ 2) Let $\sum_{i=1}^{m} \frac{1}{a_i(\cdot)} \leq \frac{1}{a(\cdot)}, \sum_{i=1}^{m} \lambda = \lambda_i, \sum_{i=1}^{m} \frac{1}{p_i(\cdot)} \leq \frac{1}{p(\cdot)}$, and $\sum_{i=1}^{m} \frac{1}{q_i} = \frac{1}{q}$ hold. Suppose that $\frac{1}{a(\cdot)} := \sum_{i=1}^{m} \frac{1}{a_i(\cdot)}$. Clearly we have $a^*(\cdot) \geq a(\cdot)$. Now take $f_i \in MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n$, where $i = 1, \ldots, m$. By utilizing the generalized Hölder’s inequality within the framework of Lebesgue spaces, it can be demonstrated that

\[
\left\| \prod_{i=1}^{m} f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n} = \sup_{L \in \mathcal{L}} \left\{ \left( \sum_{k=0}^{L} 2^{q^*(\cdot)k} \right)^{\frac{1}{q}} \right\}
\]

Taking the supremum of $2^{-L_1}$ we obtain

\[
\left\| \prod_{i=1}^{m} f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n} \leq \left\| \prod_{i=1}^{m} f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n}
\]

(2 $\Leftarrow$ 1). Suppose that $\left\| \prod_{i=1}^{m} f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n} \leq \left\| \prod_{i=1}^{m} f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n}$. For every $f_i \in MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n, i = 1, \ldots, m$. Choose $f_i := \chi_{B(0,R)}$. It can be deduced from the hypothesis that

\[
\chi_{B(0,R)} \left\| MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n \right\| \leq \prod_{i=1}^{m} \left\| f_i \right\|_{MK^{a(\cdot),\lambda}_p(\cdot),q, \mathbb{R}^n}
\]

Next, by choosing $0 < \varepsilon < \min \left\{ \frac{\alpha_1(\cdot)}{p_1(\cdot)}, \ldots, \frac{\alpha_m(\cdot)}{p_m(\cdot)} \right\}$. Clearly, $\varepsilon < d$ and for arbitrary $K \in \mathcal{H}$, write $g_{e,K}(x) := \chi (0 < |x| < 1) + \sum_{j=1}^{k} \chi (\varepsilon < j|x| < \varepsilon + \varepsilon^{-1}) (x)$ (if desired, this can be simplified to the situation where the $d = 1$, and for the general case followed by an examination of the tensor product $g_{e,K}(x_1, x_2, \ldots, x_n) = g_{e,K}(x_1), g_{e,K}(x_2), \ldots, g_{e,K}(x_n)$, working with cubes stead of balls).

By defining $f_i := g_{e,K}$, $i = 1, \ldots, m$. If $\left\| \prod_{i=1}^{m} f_i \right\|_{p(\cdot)} = g_{e,K}$ then we have $\left\| \prod_{i=1}^{m} f_i \right\|_{p(\cdot)} = g_{e,K}$. 

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Hence we obtain
\[ \left\| \prod_{i=1}^{m} f_i \right\|_{MK_N^{p(x),q(x)}} \geq C(K + K^{-\epsilon})^{1/p - 1/q} \]
\[ = (K + K^{-\epsilon})^{1/p - 1/q} \]

For each integer \( i = 1, \ldots, m \) assert that
\[ \sup_{a \in \mathbb{R}, r > 0} |B(a, r)|^{1/p - 1/q} \left( \int_{B(a,r)} f_i(x)^{\alpha_i} \, dx \right)^{1/q} \]
\[ \leq |B(0, L)|^{1/p - 1/q} \left( \int_{B(0,L)} f_i(x)^{\alpha_i} \, dx \right)^{1/q} \]

for a specific integer \( L \) with \( 2 \leq L \leq K + 1 \), it is worth mentioning that function \( f_i = g_{e,K} \) demonstrates symmetry around the origin and centralizes the majority of its distribution near 0. Consequently, for every \( a \in \mathbb{R}^n \) and \( r > 0 \), it follows that
\[ |B(a, r)|^{1/p - 1/q} \left( \int_{B(a,r)} f_i(x)^{\alpha_i} \, dx \right)^{1/q} \]
\[ \leq |B(0, r)|^{1/p - 1/q} \left( \int_{B(0,r)} f_i(x)^{\alpha_i} \, dx \right)^{1/q} \]

Indeed, the value of the bracket on the right-hand side, expressed only as a function of \( r \), demonstrates a continuous increase as \( r \) ranges from 0 to 2, contrasting with a decrease for \( r > K + K^{-\epsilon} \). This observation validates the assertion regarding the supremum. since \( \frac{1}{p_i} - \frac{1}{q_i} \leq 0 \) for \( i = 1, \ldots, m \) and \( j + j^{-\epsilon} \leq 2j \) for \( j = 1, \ldots, K \), it can be deduced that
\[ \left\| f_i \right\|_{MK_N^{p(x),q(x)}} \leq \sup_{a \in \mathbb{R}^n, r > 0} |B(a, r)|^{1/p - 1/q} \left( \int_{B(a,r)} f_i(x)^{\alpha_i} \, dx \right)^{1/q} \]
\[ \leq |B(0, L)|^{1/p - 1/q} \left( \int_{B(0,L)} f_i(x)^{\alpha_i} \, dx \right)^{1/q} \]

Furthermore, given \( L \leq K + 1 \leq 2(K + K^{-\epsilon}) \), it can be desired that
\[ \left\| f_i \right\|_{MK_N^{p(x),q(x)}} \leq C(K + K^{-\epsilon})^{1/p - 1/q} \] for \( i = 1, \ldots, m \).

In fact, \( \sum_{j=1}^{m} \frac{\alpha_i}{\alpha_i} = \frac{n}{\alpha_i} \) and \( \left\| f_i \right\|_{MK_N^{p(x),q(x)}} \leq C \), we conclude from the two inequalities above that \((K + K^{-\epsilon})^{1/p - 1/q} \leq C \) for every \( K \in \mathbb{N} \) therefore \( \sum_{j=1}^{m} \frac{\epsilon}{\alpha_i} \leq \frac{\epsilon}{\alpha_i} \) as desired.

Remark 2.2 For the cases where \( m = 2 \), the proof Theorem 2 below is derived.
Theorem 2. Let \( \alpha(\cdot), \alpha_1(\cdot), \alpha_2(\cdot) \in \mathbb{R}^n, 0 \leq \lambda, \lambda_1, \lambda_2 < \infty, p(\cdot) : p_1(\cdot), p_2(\cdot) \in P^{\mathbb{R}_n}, \) and \( 0 < q, q_1, q_2 < \infty. \) Then the subsequent statements are equivalent:

1. \( \alpha_1(\cdot) + \alpha_2(\cdot) \leq \alpha(\cdot), \lambda_1 + \lambda_2 = \lambda, \frac{1}{q_1} + \frac{1}{q_2} \leq \frac{1}{q}, \) and \( \frac{1}{p_1(\cdot)} + \frac{1}{p_2(\cdot)} \leq \frac{1}{p(\cdot)} \), and

2. \( \|f\|_{MK^{\alpha(\cdot),\lambda}(\cdot)} \leq \|f\|_{MK^{\alpha(\cdot),\lambda_1}(\cdot)} \|f\|_{MK^{\alpha(\cdot),\lambda_2}(\cdot)} \) for every \( f \in MK^{\alpha(\cdot),\lambda}(\cdot) \). The relation between \( MK^{\alpha(\cdot),\lambda}(\cdot) \) and \( MK^{p(\cdot),\lambda}(\cdot) \) can be rewritten in the following Lemma:

**Lemma 1.** Let \( 0 < q \leq \infty, \lambda \leq \infty, \alpha(\cdot) \in \mathbb{R}^n, \) and \( p(\cdot) \in P^{\mathbb{R}_n}. \) Then \( MK^{\alpha(\cdot),\lambda}(\cdot) \subseteq MK^{p(\cdot),\lambda}(\cdot) \) with \( \|f\|_{MK^{\alpha(\cdot),\lambda}(\cdot)} \leq \|f\|_{MK^{p(\cdot),\lambda}(\cdot)} \) for every \( f \in MK^{\alpha(\cdot),\lambda}(\cdot) \).

Meanwhile, our second primary finding pertains to the frail space of the generalized Hölder’s inequality with a variable exponent within Herz-Morrey spaces.

**Theorem 3.** Let \( m \geq 2. \) If \( \alpha(\cdot), \alpha_i(\cdot) \in \mathbb{R}^n, 0 \leq \lambda, \lambda_i < \infty, p(\cdot), p_i(\cdot) \in P^{\mathbb{R}_n}, \) and \( 0 \leq q, q_i < \infty, \) for each \( i = 1, \ldots, m \), then the subsequent statements are equivalent:

1. \( \sum_{i=1}^{m} \frac{1}{q_i} = 1 \) \( \lambda = \lambda_i, \sum_{i=1}^{m} \frac{1}{p_i(\cdot)} = \frac{1}{p(\cdot)} \), and \( \sum_{i=1}^{m} \frac{1}{q_i} = 1 \) hold. Suppose that \( \frac{1}{a(\cdot)} := \frac{1}{a(\cdot)} \) \( \lambda \) \( \lambda_i \), \( \frac{1}{p_i(\cdot)} \), \( \frac{1}{p(\cdot)} \). Clearly we have \( \alpha(\cdot) \geq \alpha_i(\cdot) \) \( \alpha_i(\cdot) \geq \alpha(\cdot) \) \( p_i(\cdot) \). Taking \( f_i \in MK^{\alpha_i(\cdot),\lambda_i}(\cdot), i = 1, \ldots, m \) and by using the generalized Hölder’s inequality in Lebesgue spaces, it can be demonstrated that

\[
\left\| \prod_{i=1}^{m} f_i \right\|_{MK^{\alpha_i(\cdot),\lambda}(\cdot)} \geq \frac{1}{2} \left[ B(0, K+K^-) \right]^{\frac{1}{p(\cdot)}} \frac{1}{a(\cdot)} \left( \chi \in B(a, r) : |f_i(x)| = 1 \right) \]

By leveraging Lemma 1 in conjunction with the Morrey-norm estimate pertaining to \( f_i \), it is feasible to derive

\[
\left\| \prod_{i=1}^{m} f_i \right\|_{MK^{\alpha_i(\cdot),\lambda}(\cdot)} \leq \left\| \prod_{i=1}^{m} f_i \right\|_{MK^{\alpha_i(\cdot),\lambda}(\cdot)} \leq C(K+K^-)^{\frac{1}{p(\cdot)}} \frac{1}{a(\cdot)}
\]

For each \( i \) ranging from 1 to \( m \). Given \( \sum_{i=1}^{m} \frac{1}{p_i(\cdot)} = \frac{1}{p(\cdot)} \) and

\[
\left\| \prod_{i=1}^{m} f_i \right\|_{MK^{\alpha_i(\cdot),\lambda}(\cdot)} \leq m \sum_{i=1}^{m} \left\| f_i \right\|_{MK^{\alpha_i(\cdot),\lambda}(\cdot)}
\]

it follows that \( (K+K^-)^{\frac{1}{p(\cdot)}} \sum_{i=1}^{m} \frac{1}{a_i(\cdot)} \leq C. \)

As it valid \( K \in \mathbb{N} \) for all, it can be inferred that \( \sum_{i=1}^{m} \frac{1}{a_i(\cdot)} \leq \frac{1}{a(\cdot)}. \)
4. CONCLUSION

Herz–Morrey spaces are formally defined as spaces where functions satisfy specific conditions involving variable exponents, leading to the establishment of weak Herz–Morrey spaces with variable exponents. The conditions for the generalized Hölder’s inequality with a variable exponent in Herz–Morrey spaces was investigated by providing mathematical proofs and analysis to support the results. The main Theorems established the equivalence of certain conditions for the generalized Hölder’s inequality in Herz–Morrey spaces with variable exponents, contributing to the understanding of Hölder’s inequalities in such spaces.

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